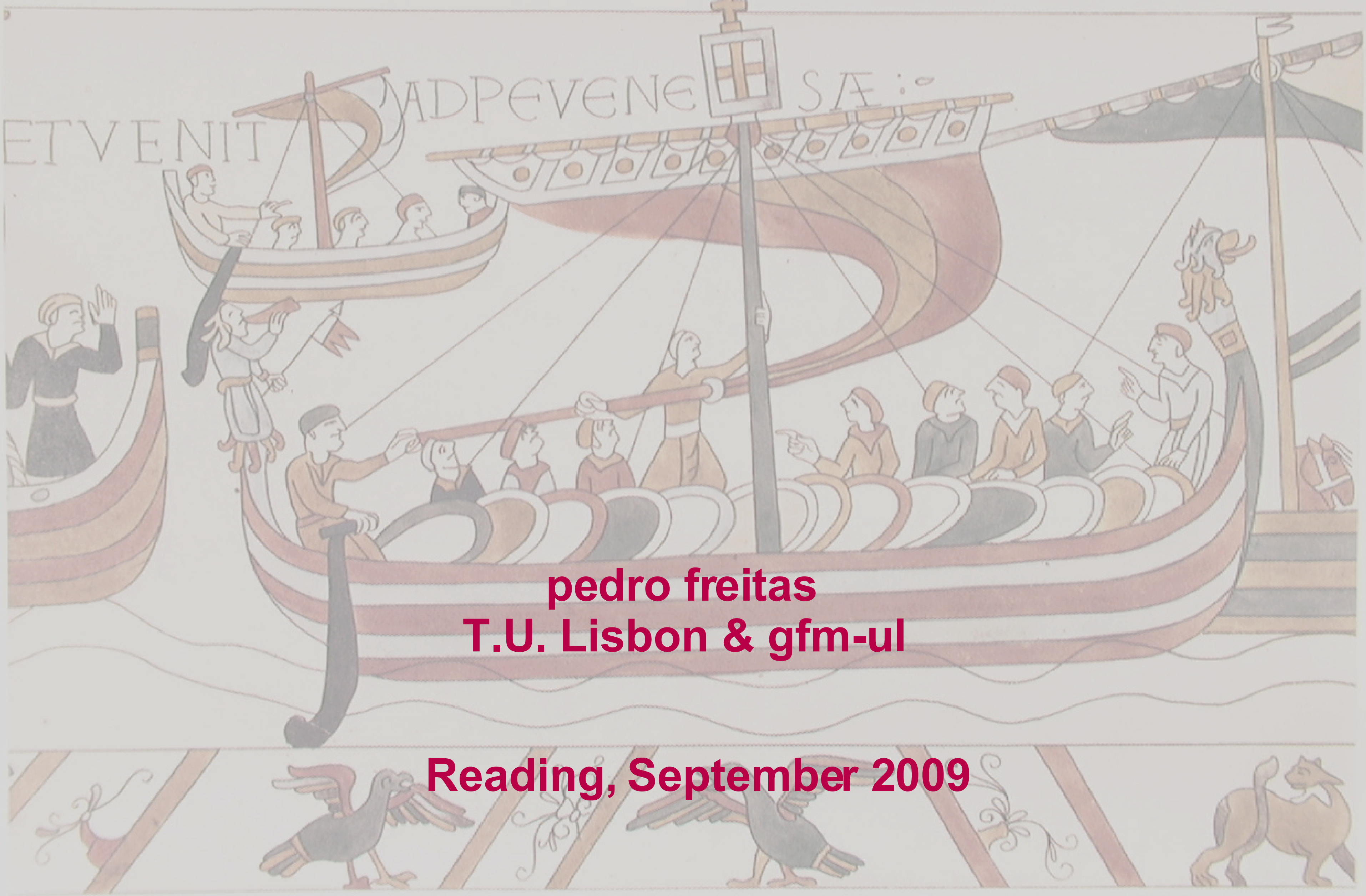
inverse problems for the damped wave equation



pedro freitas
T.U. Lisbon & gfm-ul

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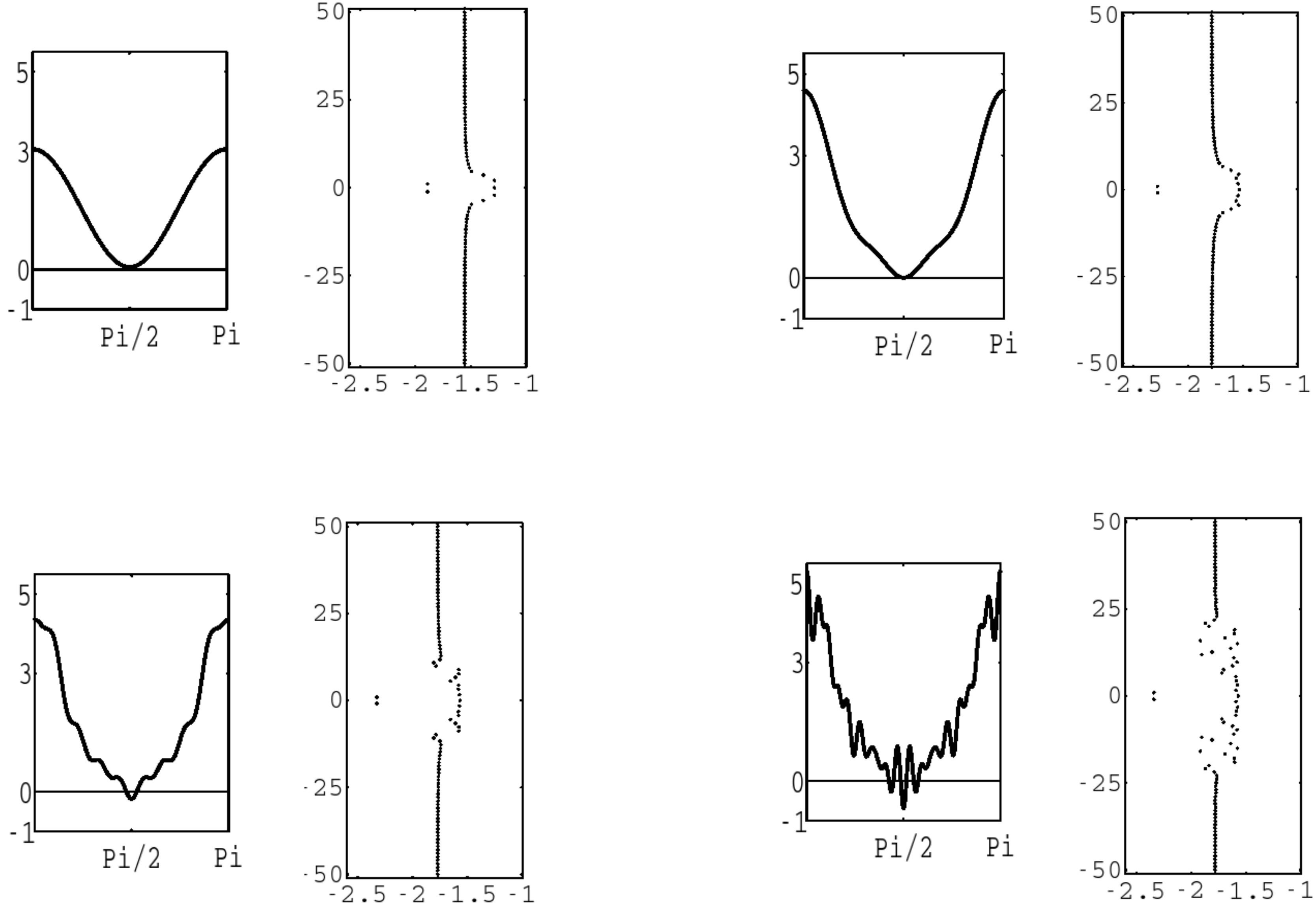
$$L(\lambda) = \lambda^2 + 2a(x)\lambda - \Delta$$


5
3
0
-1
Pi/2
Pi
50
25
0
-25
-50
-2.5 -2 -1.5 -1

Chen et al. (1991)

Cox and Zuazua (1994)

Lebeau (1994)

Cox and Knobel (1996)

F. (1996, 1999, 1999)

F. and Zuazua (1996)

F. and Lancaster (1999)

Sjöstrand (2000)

Castro and Cox (2001)

Asch and Lebeau (2003)

F. and Krejčiřík (2005)

Problems:

1. What properties do admissible spectral sequences satisfy?

2. What do isospectral sets look like?

Spectral sequences will satisfy restrictions of two types:

1. asymptotic behaviour

2. relations between eigenvalues

Other examples:

1. Sturm-Liouville

2. Dirichlet Laplacian

In the case of the wave equation (and of quadratic matrix polynomials) we may expect the restrictions to be more complicated, since now the operator is no longer self-adjoint and there exist (an infinite number of) complex eigenvalues.

Definition *Denote the eigenvalues of B and C in increasing order by b_j and c_j, $j = 1, ..., n$, respectively. Given a Hermitian pencil $L(\lambda) = I\lambda^2 + B\lambda + C$ write its eigenvalues λ_j, $j = 1, ..., 2n$, in such a way that $|\lambda_1| \leq \cdots \leq |\lambda_{2n}|$, and let $p = (p_1, ..., p_{2n})$ be a permutation of $(1, ..., 2n)$ for which $Re(\lambda_{p_1}) \geq \cdots \geq Re(\lambda_{p_{2n}})$. We define the numbers*

$$f_k = \prod_{j=1}^{k} c_j - \prod_{j=1}^{k} |\lambda_{2j-1}\lambda_{2j}|$$

and

$$g_k = \sum_{j=1}^{k} \left[b_j + Re(\lambda_{p_{2j-1}} + \lambda_{p_{2j}}) \right],$$

for $k = 1, \ldots, n$.

Theorem (Weakly damped quadratic pencils (F. 99)) *Let B (resp. C) be a Hermitian $n \times n$ matrix and Λ be a sequence of $2n$ nonreal numbers closed under complex conjugation. Then there exists a Hermitian matrix C (resp. B) such that Λ is an admissible spectral sequence for $L(\lambda) = I\lambda^2 + B\lambda + C$ if and only if the numbers g_k (resp. f_k) are nonpositive for all $k = 1, ..., n-1$ and g_n (resp. f_n) is zero.*

Theorem (Weakly damped wave equation (F. 99)) *Consider the quadratic pencil:* $L(\lambda) = \lambda^2 + 2b(x)\lambda - \Delta$ *on a bounded domain* $\Omega \in \mathbb{R}^n$*. Assume that* b *is such that* L *is weakly damped and denote its eigenvalues by* $\lambda^{\pm}$ *such that* $|\lambda_1^{\pm}| \leq |\lambda_2^{\pm}| \leq \ldots$ *and* $\lambda_k^+ = \overline{\lambda_k^-}$*. Then the numbers* f_k *are nonpositive for all* k*.*

Corollary *Let* $\Omega = (0, \pi)$*. Then*

$$\left|\lambda_k^{\pm}\right| \geq (k!)^{1/k}.$$

The inverse spectral problem for the wave equation (joint work with D. Borisov)

$$\begin{cases} (\lambda^2 + 2\lambda a - b - \partial_{xx}^2)u = 0, \ \ x \in (0,1) \\ \\ u(0) = u(1) = 0, \end{cases}$$

Denote the eigenvalues of this problem by λ_n, $n \neq 0$, and order them as follows

$$\ldots \leqslant \mathrm{Im}\,\lambda_{-2} \leqslant \mathrm{Im}\,\lambda_{-1} \leqslant \mathrm{Im}\,\lambda_1 \leqslant \mathrm{Im}\,\lambda_2 \leqslant \ldots$$

while assuming that $\lambda_{-n} = \overline{\lambda}_n$.

Theorem *Suppose* $a \in C^{m+1}[0,1]$, $b \in C^m[0,1]$, $m \geqslant 1$.

We have the following asymptotic behaviour as $n \to \pm\infty$:

$$\lambda_n = \pi n \mathrm{i} + \sum_{j=0}^{m-1} c_j n^{-j} + \mathcal{O}(n^{-m}),$$

were the c_j*'s are numbers which can be determined explicitly.*

In particular,

$$c_0 = -\langle a \rangle, \quad c_1 = \frac{\langle a^2 + b \rangle}{2\pi \mathrm{i}},$$
$$c_2 = \frac{1}{2\pi^2}\left[\langle a(a^2+b)\rangle - \langle a \rangle \langle a^2 + b \rangle + \frac{a'(1) - a'(0)}{2}\right].$$

Corollary *Assume that* $a \in C^3[0,1]$ *and* $b \in C^2[0,1]$ *is fixed. Then the function* $a(x)$ *is constant if and only if*

$$c_0^2 = 2\pi \mathrm{i} c_1 - \langle b \rangle,$$

in which case $a(x) \equiv -c_0$.

Theorem *Let* $a \in C^3[0,1]$, $b \in C^2[0,1]$. *Then the identity*

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (\lambda_n - c_0) = \frac{a(0)+a(1)}{2} - \langle a \rangle$$

holds.

Ideas of the proof:

(Follows very closely the proof of similar results for the Sturm-Liouville problem)

1. Consider solutions of the form

$$u_\pm(x,\lambda) = \mathrm{e}^{\pm\lambda x \pm \int_0^x \phi_\pm(t,\lambda)\,\mathrm{d}t}$$

$$\text{with } \phi_\pm(x,\lambda) = \sum_{i=0}^{m} \phi_i^{(\pm)}(x)\lambda^{-i} + \mathcal{O}(\lambda^{-m-1}), \quad m \geqslant 1.$$

2. Write

$$u(x,\lambda) = \frac{u_+(x,\lambda) - u_-(x,\lambda)}{u'_+(0,\lambda) - u'_-(0,\lambda)}$$

3. Use a shooting method.