# Spectral problems related to a time dependent model in superconductivity with electric current. (after Almog, Almog-Helffer-Pan, . . . )

Bernard Helffer (Univ Paris-Sud and CNRS)

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This talk was initially motivated by a paper of Yaniv Almog at Siam [Alm2], one chapter of a book of Brian Davies, some chapter of the PHD of my student J. Martinet, some chapters of a course of Embree-Trefethen and discussions with F. Nier, F. Hérau and W. Bordeaux-Montrieux. The main goal is to show how the pseudo-spectrum of some non self-adjoint operators appear in a specific problem appearing in superconductivity, for which we have obtained recently results together with Y. Almog and X. Pan [AlmHelPan1, AlmHelPan2].
After a presentation of the general problems and of our main results, we will come back to Almog's analysis, consider the complex Airy operator on $D_x^2 + ix$ on the line or on $\mathbb{R}^+$ and make a survey of what is known. If time permit, we will also mention recent results obtained in collaboration with J. Sjöstrand.

Consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state.

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To be more precise, let us consider a two-dimensional superconducting sample capturing the entire $xy$ plane. We can assume also that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample. Denote the Ginzburg-Landau parameter of the superconductor by $\kappa$ and the normal conductivity of the sample by $\sigma$.

The physical problem is posed in a domain $\Omega$ with specific boundary conditions.
We will only analyze limiting situations where the domains possibly after a blowing argument become the whole space (or the half-space).
We will mainly work in $2D$ for simplification. $3D$ is also very important.

Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in $(0, T) \times \mathbb{R}^2$ :

$$\begin{cases} \partial_t \psi + i\,\kappa\,\Phi\psi = \nabla^2_{\kappa\mathbf{A}}\psi + \kappa^2\,(1 - |\psi|^2)\psi\,, \\ \kappa^2 \ \text{curl}\ {}^2\mathbf{A} + \sigma\,(\partial_t\mathbf{A} + \nabla\Phi) = \kappa\,\mathrm{Im}\,(\bar{\psi}\nabla_{\kappa\mathbf{A}}\psi) + \kappa^2\ \text{curl}\ \mathcal{H}^e\,, \end{cases} \tag{1}$$

where $\psi$ is the order parameter, $\mathbf{A}$ is the magnetic potential, $\Phi$ is the electric potential, and $(\psi, \mathbf{A}, \Phi)$ also satisfies an initial condition at $t = 0$.

In order to solve this equation, one should also define a gauge (Coulomb, Lorenz,...). The orbit of $(\psi, \mathbf{A}, \Phi)$ is $(\exp i\kappa q\, \psi, \mathbf{A} + \nabla q, \Phi - \partial_t q)$ where $q$ is a function of $(x, t)$. We refer to [BaJaPhi] for a discussion of this point. We will choose the Coulomb gauge which reads that we can add the condition $\operatorname{div} \mathbf{A} = 0$ for any $t$. A solution $(\psi, \mathbf{A}, \Phi)$ is called a normal state solution if $\psi = 0$.

# Stationary normal solutions

From (1) we see that if $(0, \mathbf{A}, \Phi)$ is a time-independent normal state solution then $(\mathbf{A}, \Phi)$ satisfies the equality

$$\kappa^2 \text{ curl }^2\mathbf{A} + \sigma\nabla\Phi = \kappa^2 \text{ curl } \mathcal{H}^e\,, \;\; \text{div } \mathbf{A} = 0 \quad \text{in } \mathbb{R}^2\,. \qquad (2)$$

(Note that if one identifies $\mathcal{H}^e$ to a function $h$, then curl $\mathcal{H}^e = (-\partial_y h\,,\, \partial_x h)$.)

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(Note that if one identifies $\mathcal{H}^e$ to a function $h$, then $\text{curl } \mathcal{H}^e = (-\partial_y h\,,\, \partial_x h)$.)

This could be rewritten as the property that

$$\kappa^2(\text{ curl } \mathbf{A} - \mathcal{H}^e) + i\sigma\Phi\,,$$

is an holomorphic function.

In particular

$$\Delta\Phi = 0 \text{ and } \Delta(\text{ curl } \mathbf{A} - \mathcal{H}^e)) = 0\,.$$

# Special situation: $\Phi$ affine

(1) has the following stationary normal state solution

$$\mathbf{A} = \frac{1}{2J}(Jx + h)^2 \hat{\imath}_y , \quad \Phi = \frac{\kappa^2 J}{\sigma} y \,. \tag{3}$$

Note that

$$\operatorname{curl} \mathbf{A} = (Jx + h)\, \hat{\imath}_z ,$$

that is, the induced magnetic field equals the sum of the applied magnetic field $h\hat{\imath}_z$ and the magnetic field produced by the electric current $Jx\, \hat{\imath}_z$.

For this normal state solution, the linearization of (1) with respect to the order parameter is

$$\partial_t \psi + \frac{i\kappa^3 J y}{\sigma}\psi = \Delta\psi - \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi\,. \tag{4}$$

Applying the transformation $x \to x - J/h$, the time-dependent linearized Ginzburg-Landau equation takes the form

$$\frac{\partial\psi}{\partial t} + i\frac{J}{\sigma}y\psi = \Delta\psi - iJx^2\frac{\partial\psi}{\partial y} - \left(\frac{1}{4}J^2x^4 - \kappa^2\right)\psi\,. \tag{5}$$

Rescaling $x$ and $t$ by applying

$$t \to J^{2/3}t \; ; \; (x,y) \to J^{1/3}(x,y)\,, \tag{6}$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda)u\,, \tag{7}$$

where

$$\mathcal{A}_{0,c} := D_x^2 + (D_y - \frac{1}{2}x^2)^2 + icy\,, \tag{8}$$

and

$$c = 1/\sigma \; ; \; \lambda = \frac{\kappa^2}{J^{2/3}} \; ; \; u(x,y,t) = \psi(J^{-1/3}x, J^{-1/3}y, J^{-2/3}t)\,.$$

Our main problem will be to analyze the long time property of the attached semi-group.

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We now apply the transformation

$$u \to u\, e^{icyt}$$

to obtain

$$\partial_t u = -\left(D_x^2 u + (D_y - \frac{1}{2}x^2 - ct)^2 u - \lambda u\right) . \tag{9}$$

Note that considering the partial Fourier transform, we obtain

$$\partial_t \hat{u} = -D_x^2 \hat{u} - \left[\left(\frac{1}{2}x^2 + (ct - \omega)\right)^2 - \lambda\right] \hat{u} \,. \tag{10}$$

This can be rewritten as a family (depending on $\omega \in \mathbb{R}$) of time-dependent problems on the line

$$\partial_t \hat{u} = -\mathcal{L}_{\beta(t,\omega)} \hat{u} + \lambda \hat{u} \,, \tag{11}$$

with $\mathcal{L}_\beta$ being the well-known anharmonic oscillator (or Montgomery operator) :

$$\mathcal{L}_\beta = D_x^2 + (\frac{1}{2}x^2 + \beta)^2 \,, \tag{12}$$

and

$$\beta(t,\omega) = ct - \omega \,.$$

Note that in this change of point of view, we can after a change of time look at the family of problems

$$\partial_\tau v(x,\tau) = -(\mathcal{L}_{c\tau} v)(x,\tau) + \lambda v(x,\tau)\,, \tag{13}$$

the initial condition at $t = 0$ becoming an initial condition at $\tau = -\frac{\omega}{c}$.

# Recent results by Almog-Helffer-Pan

If $c \neq 0$, $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ has compact resolvent, empty spectrum, and

$$\|\exp(-t\mathcal{A})\| \leq \exp\left(-\frac{2\sqrt{2c}}{3} t^{3/2} + C t^{3/4}\right). \tag{14}$$

There exists $C$ such that, for all $\lambda$ such that $\mathrm{Re}\,\lambda \geq 0$,

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \exp\left(\frac{1}{6c} \mathrm{Re}\,\lambda^3 + C\,\mathrm{Re}\,\lambda^{3/2}\right). \tag{15}$$

We have also a lower bound of the same type using a quasimode construction.

# Simplified model: no magnetic field

We assume, following Almog, that a current of constant magnitude $J$ is being flown through the sample in the $x$ axis direction, and $h = 0$.

Then (1) has (in some asymptotic regime) the following stationary normal state solution

$$\mathbf{A} = 0\,, \quad \Phi = Jx\,. \tag{16}$$

For this normal state solution, the linearization of (1) gives

$$\partial_t \psi + iJx\psi = \Delta_{x,y}\psi + \psi\,, \tag{17}$$

whose analysis is strongly related to the Airy equation.

# The Airy operator in $\mathbb{R}$

The operator $D_x^2 + ix$ can be defined as the closed extension $\mathcal{A}$ of the differential operator on $C_0^\infty(\mathbb{R})$ :

$$\mathcal{A}_0^+ := D_x^2 + ix\,. \tag{18}$$

We observe that

$$\mathcal{A} = (\mathcal{A}_0^-)^* \quad \text{with } \mathcal{A}_0^- := D_x^2 - ix\,. \tag{19}$$

and,

$$D(\mathcal{A}) = \{u \in H^2(\mathbb{R})\,,\, xu \in L^2(\mathbb{R})\}\,. \tag{20}$$

In particular $\mathcal{A}$ has compact resolvent.

It is also easy to see that

$$\mathrm{Re}\,\langle \mathcal{A}u \mid u\rangle \geq 0\,. \tag{21}$$

Hence $-\mathcal{A}$ is the generator of a semi-group $S_t$ of contraction,

$$S_t = \exp -t\mathcal{A}\,. \tag{22}$$

Hence all the results of this theory can be applied.
In particular, we have, for $\mathrm{Re}\,\lambda < 0$

$$||(\mathcal{A}-\lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,. \tag{23}$$

One can also show that the operator is maximally accretive.

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a \mathcal{A} = (\mathcal{A} - ia) T_a \,, \tag{24}$$

where $T_a$ is the translation operator :

$$(T_a u)(x) = u(x-a)\,. \tag{25}$$

As immediate consequence, we obtain that the spectrum is empty

$$\sigma(\mathcal{A}) = \emptyset \tag{26}$$

and that the resolvent of $\mathcal{A}$, which is defined for any $\lambda \in \mathbb{C}$ satisfies

$$||(\mathcal{A} - \lambda)^{-1}|| = ||(\mathcal{A} - \mathrm{Re}\,\lambda)^{-1}||\,. \tag{27}$$

The most interesting property is the control of the resolvent for $\mathrm{Re}\,\lambda \geq 0$.

**Proposition (W. Bordeaux-Montrieux [BM])**

As $\mathrm{Re}\,\lambda \to +\infty$, we have

$$||(A-\lambda)^{-1}|| \sim \sqrt{\frac{\pi}{2}}|\mathrm{Re}\,\lambda|^{-\frac{1}{4}} \exp \frac{4}{3}\mathrm{Re}\,\lambda^{\frac{3}{2}}\,, \tag{28}$$

A weaker result was previously obtained by Martinet [Mart]. Many results of this type are obtained in Trefethen-Embree's book for other model but this one is extremely accurate.

The proof of the (rather standard) upper bound is based on the direct analysis of the semi-group in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + ix)\mathcal{F}^{-1} = \xi^2 + \frac{d}{d\xi}\,. \tag{29}$$

Then we have

$$\mathcal{F} S_t \mathcal{F}^{-1} v = \exp(-\xi^2 t + \xi t - \frac{t^3}{3}) v(\xi - t)\,, \tag{30}$$

and this implies immediately

$$||S_t|| = \exp \max_{\xi}(-\xi^2 t + \xi t - \frac{t^3}{3}) = \exp(-\frac{t^3}{12})\,. \tag{31}$$

Note that this implies that the spectrum of $\mathcal{A}$ is empty.

Then one can get an estimate of the resolvent by using, for $\lambda$ real, the formula

$$(\mathcal{A}-\lambda)^{-1}=\int_0^{+\infty}\exp -t(\mathcal{A}-\lambda)\,dt\,. \tag{32}$$

We immediately obtain

$$||(\mathcal{A}-\lambda)^{-1}||\leq\int_0^{+\infty}\exp(\lambda t-\frac{t^3}{12})\,dt\,. \tag{33}$$

The asymptotic behavior as $\lambda\to+\infty$ of this integral is immediately obtained by using the Laplace method and the dilation $t=\lambda^{\frac{1}{2}}s$ in the integral.

The proof of the lower bound is obtained by constructing quasimodes for the operator $(\mathcal{A} - \lambda)$ in its Fourier representation.
We observe that

$$u(\xi;\lambda) := \exp -\frac{\xi^3}{3} + \lambda\xi \tag{34}$$

is a solution of

$$(\partial_\xi + \xi^2 - \lambda)u(\xi;\lambda) = 0\,. \tag{35}$$

Multiplying $u$ by a cut-off function in such a way that at the cut-off $u$ is relatively much smaller than at its maximum, we obtain a very good quasimode with an error term giving the stated lower bound for the resolvent.
Of course this is a very special case of a result on the pseudo-spectrum but this leads to an almost optimal result. The optimal result is obtained by introducing a Grushin's problem.

# The Airy complex operator in $\mathbb{R}^+$

Here we mainly describe some results presented in [Alm2], who refers to [IvKol]. We can then associate the Dirichlet realization $\mathcal{A}^D$ of $D_x^2 + ix$ on the half space. Again we have an operator, which is the generator of a semi-group of contraction, whose adjoint is described by replacing in the previous description $(D_x^2 + ix)$ by $(D_x^2 - ix)$, the operator is injective and as its spectrum contained in $\mathrm{Re}\,\lambda > 0$. Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation, we obtain ([Alm2]) that

$$\sigma(\mathcal{A}^D) = \cup_{j=1}^{+\infty}\{\lambda_j\} \tag{36}$$

with

$$\lambda_j = \exp i\frac{\pi}{3}\,\mu_j\,, \tag{37}$$

the $\mu_j$'s being real zeroes of the Airy function satisfying

$$0 < \mu_1 < \cdots < \mu_j < \mu_{j+1} < \cdots\,. \tag{38}$$

It is also shown in [Alm2] that the vector space generated by the corresponding eigenfunctions is dense in $L^2(\mathbb{R}^+)$. But there is no way to normalize these eigenfunctions for getting a good basis of $L^2(\mathbb{R}^+)$. See Almog, Davies and the master thesis of R. Henry. Following Davies, we say in this case that $\mathcal{A}^D$ is spectrally wild.

We arrive now to the analysis of the properties of the semi-group and the estimate of the resolvent.
As before, we have, for $\mathrm{Re}\,\lambda < 0$,

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \tag{39}$$

If $\mathrm{Im}\,\lambda < 0$ one gets also a similar inequality, so the main remaining question is the analysis of the resolvent in the set $\mathrm{Re}\,\lambda \geq 0\,,\ \mathrm{Im}\,\lambda \geq 0$, which corresponds to the numerical range of the symbol.

We recall that for any $\epsilon > 0$, we define the $\epsilon$-pseudospectrum by

$$\Sigma_\epsilon(\mathcal{A}^D) = \{\lambda \in \mathbb{C} \,|\, ||(\mathcal{A}^D - \lambda)^{-1}|| > \frac{1}{\epsilon}\}\,, \tag{40}$$

with the convention that $||(\mathcal{A}^D - \lambda)^{-1}|| = +\infty$ if $\lambda \in \sigma(\mathcal{A}^D)$.
We have

$$\cap_{\epsilon>0}\Sigma_\epsilon(\mathcal{A}^D) = \sigma(\mathcal{A}^D)\,. \tag{41}$$

We define, for any accretive operator, for $\epsilon > 0$,

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \Sigma_\epsilon(\mathcal{A})} \mathrm{Re}\, z\,. \tag{42}$$

We also define

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t \to +\infty} \frac{1}{t} \log || \exp -t\mathcal{A}|| \tag{43}$$

$$\widehat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z \in \sigma(\mathcal{A})} \mathrm{Re}\, z\,. \tag{44}$$

## Theorem (Gearhart-Prüss)

Let $\mathcal{A}$ be a densely defined closed operator in an Hilbert space $X$ such that $-\mathcal{A}$ generates a contraction semi-group and let $\widehat{\alpha}_\epsilon(\mathcal{A})$ and $\widehat{\omega}_0(\mathcal{A})$ denote the $\epsilon$-pseudospectral abcissa and the growth bound of $\mathcal{A}$ respectively. Then

$$\lim_{\epsilon\to 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A})\,. \tag{45}$$

We refer to [EngNag] for a proof. A more quantitative version is proposed in [HelSj].

This theorem is interesting because it reduces the question of the decay, which is basic in the question of the stability to an analysis of the $\epsilon$-spectrum of the operator.

We apply this theorem to our operator $\mathcal{A}_D$ and our main theorem is

**Theorem**

$$\widehat{\omega}_0(\mathcal{A}_D) = -\mathrm{Re}\,\lambda_1\,. \tag{46}$$

This statement was established by Almog in a much weaker form.
Using the first eigenfunction it is easy to see that

$$||\exp -t\mathcal{A}^D|| \geq \exp -\mathrm{Re}\,\lambda_1\,t\,. \tag{47}$$

Hence we have immediately

$$0 \geq \widehat{\omega}_0(\mathcal{A}^D) \geq -\mathrm{Re}\,\lambda_1\,. \tag{48}$$

To prove that $-\mathrm{Re}\,\lambda_1 \geq \widehat{\omega}_0(\mathcal{A}^D)$, it is enough to show the following lemma.

**Lemma**

For any $\alpha < \mathrm{Re}\,\lambda_1$ , there exists a constant $C$ such that, for all $\lambda$ s.t. $\mathrm{Re}\,\lambda \leq \alpha$

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq C\,. \tag{49}$$

**Proof**

We know that $\lambda$ is not in the spectrum. Hence the problem is just a control of the resolvent as $|\mathrm{Im}\,\lambda| \to +\infty$. The case, when $\mathrm{Im}\,\lambda < 0$ has already be considered. Hence it remains to control the norm of the resolvent as $\mathrm{Im}\,\lambda \to +\infty$ and $\mathrm{Re}\,\lambda \in [-\alpha, +\alpha]$.

This is indeed a semi-classical result ! The main idea is that when $\mathrm{Im}\,\lambda \to +\infty$, we have to inverse the operator

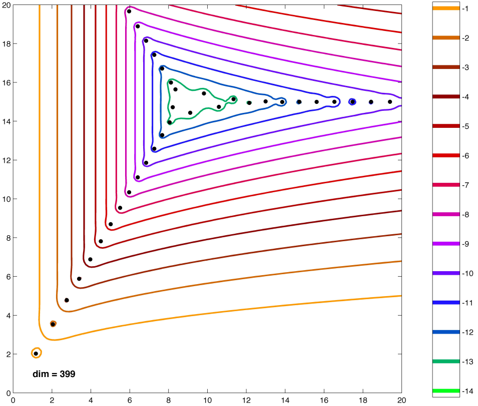
$$D_x^2 + i(x - \mathrm{Im}\,\lambda) - \mathrm{Re}\,\lambda\,.$$

If we consider the Dirichlet realization in the interval $]0, \frac{\mathrm{Im}\,\lambda}{2}[$ of $D_x^2 + i(x - \mathrm{Im}\,\lambda) - \mathrm{Re}\,\lambda$, it is easy to see that the operator is invertible by considering the imaginary part of this operator and that this inverse $R_1(\lambda)$ satisfies

$$||R_1(\lambda)|| \leq \frac{2}{\mathrm{Im}\,\lambda}.$$

Far from the boundary, we can use the resolvent of the problem on the line for which we have a uniform control of the norm for $\mathrm{Re}\,\lambda \in [-\alpha, +\alpha]$.

Figure: Airy with Dirichlet condition : pseudospectra



**Application**

Coming back to the application in superconductivity, one is looking at the semigroup associated with $\mathcal{A}_J := D_x^2 + iJx - 1$ (where $J \geq 0$ is a parameter). The stability analysis leads to a critical value $J_c$ with

$$J_c = (\mathrm{Re}\,\lambda_1)^{-\frac{3}{2}}\,. \tag{50}$$

For $J \in [0, J_c[$ the associated operator $\exp -t\mathcal{A}_J$ satisfies $||\exp -t\mathcal{A}_J|| \to +\infty$ as $t \to +\infty$.
For $J > J_c$, $||\exp -t\mathcal{A}_J|| \to 0$.
This improves Lemma 2.4 in Almog [Alm2], who gets only this decay for $||\exp -t\mathcal{A}_J\psi||$ for $\psi$ in a specific dense space.

# Higher dimension problems relative to Airy

Here we follow (and extend) [Alm2] Almog. We consider the operator

$$\mathcal{A}_2 := -\Delta_{x,y} + ix\,. \tag{51}$$

## Proposition

$$\sigma(\mathcal{A}_2) = \emptyset\,. \tag{52}$$

# The model in $\mathbb{R}^2_+$ : perpendicular current.

Here it is useful to reintroduce the parameter $J$ which is assumed to be positive. Hence we consider the Dirichlet realization

$$\mathcal{A}_2^{D,\perp} := -\Delta_{x,y} + iJx\,, \tag{53}$$

in $\mathbb{R}^2_+ = \{x > 0\}$.

**Proposition**

$$\sigma(\mathcal{A}_2^{D,\perp}) = \cup_{r\geq 0, j\in\mathbb{N}^*}(\lambda_j + r)\,. \tag{54}$$

Then we can get the following information on the $L^\infty$-spectrum.

## Proposition

Let $\Psi \in L^\infty(\mathbb{R}^2_+) \cap H^1_{loc}(\mathbb{R}^2_+)$ satisfying, for some $\lambda \in \mathbb{C}$,

$$-\Delta_{x,y}\Psi + iJx\Psi = \lambda\Psi \tag{55}$$

in $\mathbb{R}^2_+$ and

$$\Psi_{x=0} = 0\,. \tag{56}$$

Then either $\Psi = 0$ or $\lambda \in \sigma(\mathcal{A}_3^{D,\perp})$.

# The model in $\mathbb{R}_2^+$ : parallel current.

Here the models are the Dirichlet realization in $\mathbb{R}^2_+$ :

$$\mathcal{A}_2^{D,\parallel} = -\Delta_{x,y} + iJy\,, \tag{57}$$

or the Neumann realization

$$\mathcal{A}_2^{N,\parallel} = -\Delta_{x,y} + iJy\,. \tag{58}$$

Using the reflexion (or antireflexion) trick we can see the problem as a problem on $\mathbb{R}^2$ restricted to odd (resp. even) functions with respect to $(x,y) \mapsto (-x,y)$. It is clear from Proposition 1 that in this case the spectrum is empty.

# Maximal accretivity

All the operators can be placed in the following more general context. We consider in $\mathbb{R}^n$ (or in an open set)

$$-\Delta_A + V\,, \tag{59}$$

with

$$\mathrm{Re}\, V \geq 0 \ \text{ and } \ V \in C^\infty(\mathbb{R}^n)\,.$$

Then the operator is maximally accretive.

## About the $L^\infty$-spectrum

The question we are analyzing can be reformulated as a comparison between the $L^\infty$-spectrum and the $L^2$-spectrum. Assuming that we have determined the $L^2$-spectrum in the previous proposition, we would like to analyze the consequence of an admissible pair (with bounded $\psi$).

## Proposition

We assume that $V \in C^0$ and $\mathrm{Re}\, V \geq 0$ If $(\psi, \lambda)$ is an admissible $L^\infty$ pair for $P_{\mathbf{A},V}$, i.e. if

$$(P_{\mathbf{A},V} - \lambda)\psi = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n)\,,$$

with $\psi \in L^\infty$ (or $L^2$), then $\lambda$ is in the spectrum of $\mathcal{P} = \overline{P_{A,V}}$.

The proof is reminiscent of the so-called Schnol's theorem, actually due to Glazman.

# Recent results with Y. Almog and X. Pan

Coming back to our initial model we can analyze $\frac{1}{t}\ln||\exp -t\overline{P_{\mathbf{A},V}}||$, as $t \to +\infty$ for our specific examples

1. In the case of the whole space [AlmHelPan1], we have:

$$\frac{1}{t}\ln||\exp -t\overline{P_{\mathbf{A},V}}|| = -\infty\,.$$

2. In the case of the half space [AlmHelPan2], for $|c|$ large enough, we have:

$$\frac{1}{t}\ln||\exp -tP^D_{\mathbf{A},V}|| = -\inf \mathrm{Re}\,\sigma(P^D_{\mathbf{A},V}) > -\infty\,.$$

In the second case, it is actually not so easy to show that the spectrum is non empty.

# Recent results with J. Sjöstrand

The aim is to get in some cases explicit constants in order to treat problems with parameters. Let $\mathcal{H}$ be a complex Hilbert space and let $[0,+\infty[\ni t \mapsto S(t) \in \mathcal{L}(\mathcal{H},\mathcal{H})$ be a strongly continuous semigroup with $S(0) = I$. Recall that by the Banach-Steinhaus theorem, there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ s. t. $S(t)$ has the property

$$P(M,\omega_0): \quad \|S(t)\| \leq Me^{\omega_0 t},\ t \geq 0.$$

Let $A$ be the generator of the semigroup (so that formally $S(t) = \exp tA$) and recall that $A$ is closed and densely defined.

The main result is:

**Theorem**

We assume that, for some $\omega \in \mathbb{R}$, $r(\omega) > 0$ with

$$\frac{1}{r(\omega)} = \sup_{\mathrm{Re}\, z \geq \omega} ||(z - A)^{-1}|| \,.$$

Let $m(t) \geq ||S(t)||$ be a continuous positive function.
Then for all $t, a, \widetilde{a} > 0$, such that $t = a + \widetilde{a}$, we have

$$||S(t)|| \leq \frac{e^{\omega t}}{\Theta(a, \widetilde{a}, \omega, t)} \tag{60}$$

with

$$\Theta(a,\widetilde{a},\omega,t) = r(\omega)||\frac{1}{m}||_{e^{-\omega}\ L^2(]0,a[)}\ ||\frac{1}{m}||_{e-\omega\cdot\ L^2(]0,\widetilde{a}[)}\,.$$

Here the norms are always the natural ones obtained from $\mathcal{H}$, $L^2$.
In (60) we also have the natural norm in the exponentially weighted space $e^{-\omega \cdot} L^2([0,a])$ and similarly with $\widetilde{a}$ instead of $a$;
$||f||_{e^{-\omega \cdot} L^2(]0,a[)} = ||e^{\omega \cdot} f(\cdot)||_{L^2(]0,a[)}$.
As we shall see in the next section, under the assumption of the theorem, we have $P(M,\omega)$ with an explicit $M$.

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