Questions on polynomial operator pencils in connection with the non hypoanalyticity of some Hörmander's operators

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The goal is to determine, for a given homogeneous polynomial P of n variables, if there exists a complex λ such that

$$-\Delta + (P(x) - \lambda)^2$$

is non injective in $\mathcal{S}(\mathbb{R}^n)$.
This question arises at the beginning of the eighties at a period where the researchs in hypoanalyticity of pseudo-differential operators with multiple characteristics (Boutet de Montvel, Métivier,..) were very active in France and in the United States (around F. Trèves and his School (Hanges, Himonas, Kostin,.., Chanillo) and M. Christ). The problem above occurs in the proof of non-hypoanalyticity.

The initial model, proposed by B. Helffer (1979) and solved by Pham The Lai and Robert (1980), being the operator :

$$D_x^2 + (D_y - x^2 D_z)^2$$

in $\mathbb{R}^3$.
This leads to solve the following problem of finding a complex β such that

$$D_x^2 + (x^2 - \beta)^2$$

has a non trivial kernel in $\mathcal{S}(\mathbb{R})$.
More recently papers by Chanillo-Helffer-Laptev (2004) (based on the use of Lidskii's theorem) , Helffer-Robert-Wang (2005), Robert (2007), Aboud (PHD in Nantes), and Aboud-Robert (2009) have given more recent insights in the problem.

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