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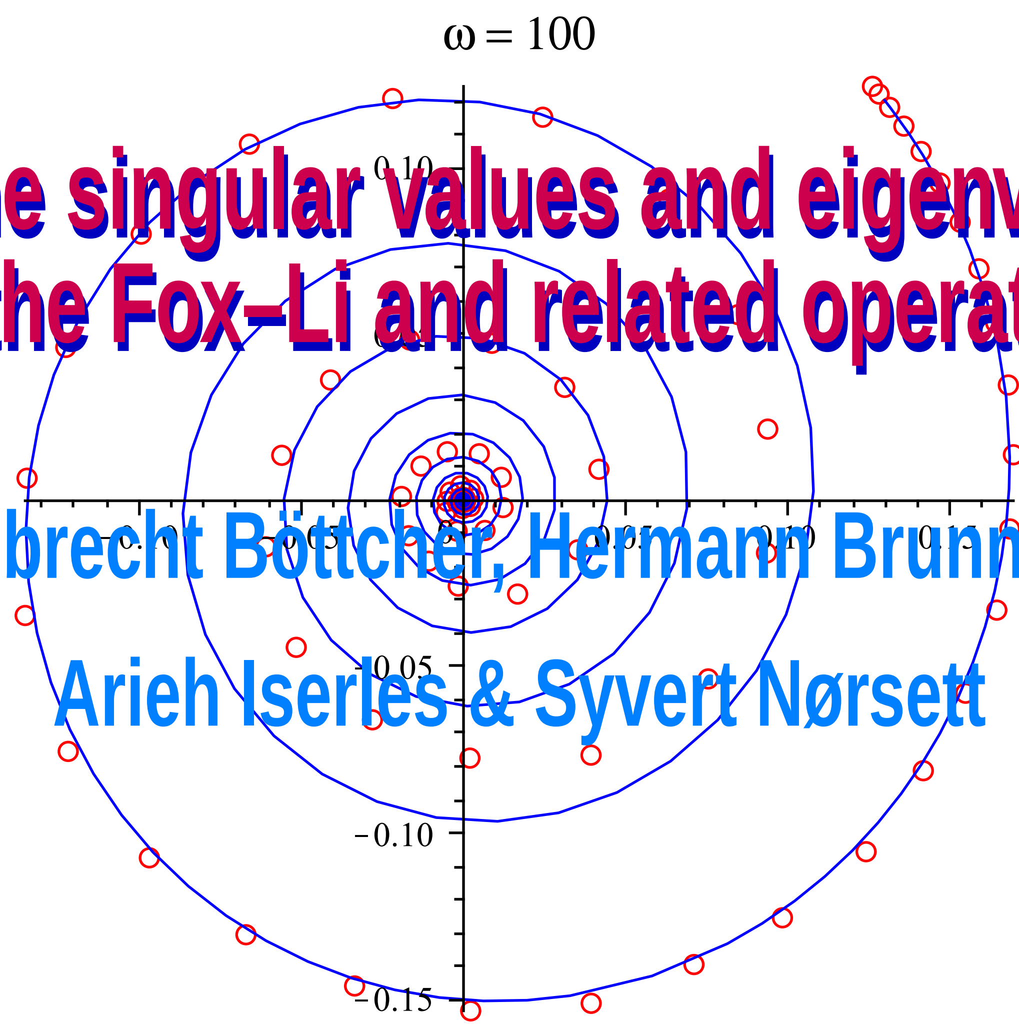

On the singular values and eigenvalues of the Fox–Li and related operators

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THE FOX–LI OPERATOR

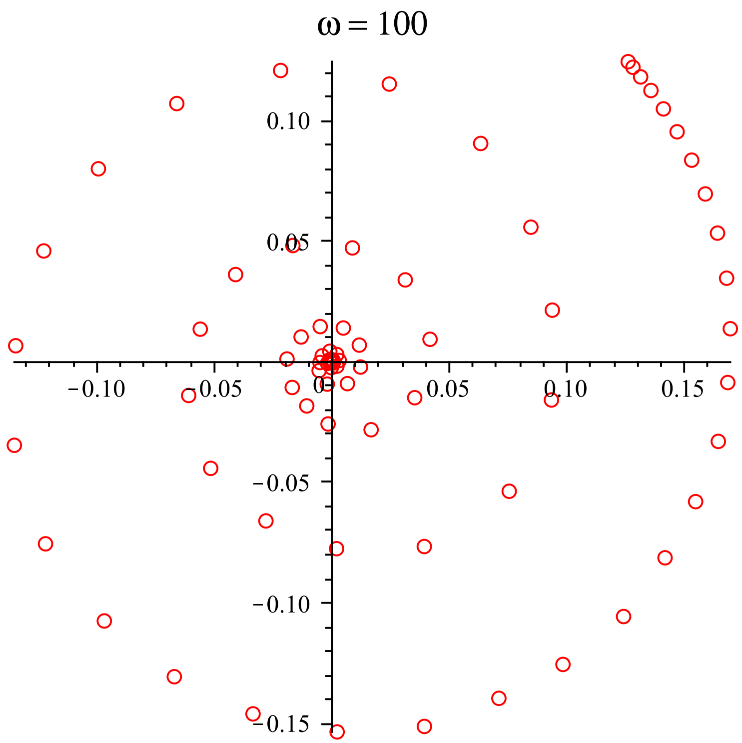
$$(\mathcal{F}_\omega f)(x) := \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} f(y)\,\mathrm{d}y, \qquad x \in (-1,1),$$

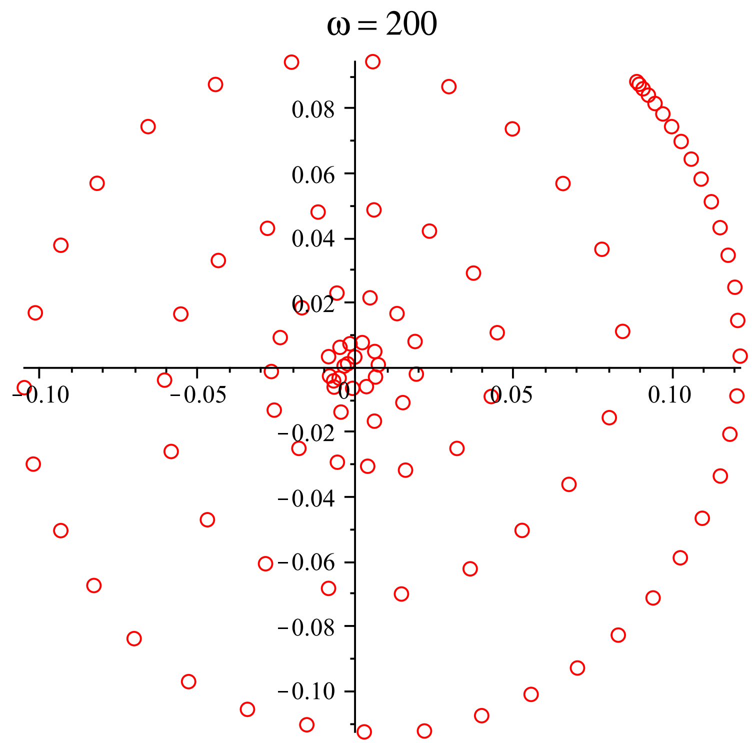
where $\omega > 0$:

- $\mathcal{F}_\omega$ is a complex-valued, symmetric linear operator in $L^2(-1,1)$. It is compact, therefore $\sigma(\mathcal{F}_\omega)$ consists of the origin and at most a countable number of eigenvalues accumulating at the origin.

- It spectrum $\sigma(\mathcal{F}_\omega)$ is important in laser and maser engineering: the eigenfunctions represent modes (self-reproducing field distributions) between two semi-circular reflectors placed at a fixed distance from each other.

- Computational results indicate that $\sigma(\mathcal{F}_\omega)$ lies on a spiral commencing near $\sqrt{\pi/\omega}\mathrm{e}^{\mathrm{i}\pi/4}$ and rotating clockwise to the origin. Not much is known of the precise shape of this spiral.

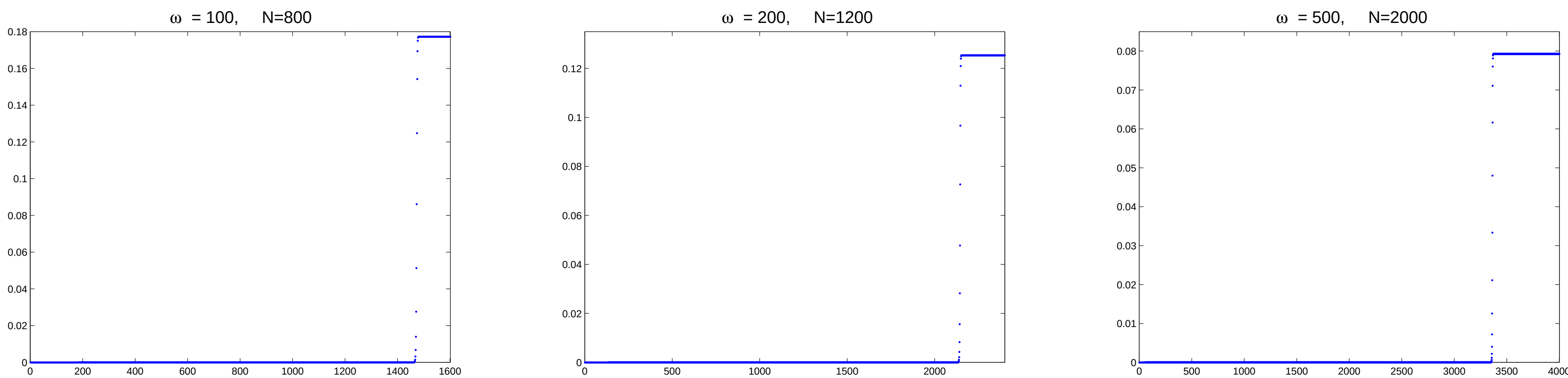




Fox–Li eigenvalues for $\omega = 100$ and $\omega = 200$.

FOX–LI SINGULAR VALUES

Singular values $s(\mathcal{F}_\omega)$ are the positive square roots of the the eigenvalues of the positive semidefinite operator $\mathcal{F}_\omega^* \mathcal{F}$. They are of an independent interest, e.g. in random matrix theory.



Fox–Li singular values, approximated as eigenvalues of a $(2N+1) \times (2N+1)$ matrix.

We observe that *almost* all singular values accumulate at just two points: the origin and something which suspiciously looks like $\sqrt{\pi/\omega}$.

WIENER–HOPF OPERATORS

The Fourier–Plancharel transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

$$\widehat{f}(\xi) = (Ff)(\xi) := \int_{-\infty}^{\infty} f(t) \mathrm{e}^{\mathrm{i}\xi t}\, \mathrm{d}t, \quad \xi \in \mathbb{R}.$$

Few definitions:

Multiplication: $M(a) : f \mapsto af;$

Convolution: $C(a) : f \mapsto F^{-1}M(a)Ff;$

Projection: P_+ orthogonally projects $L^2(\mathbb{R}) \to L^2(0,\infty)$,
P_τ orthogonally projects $L^2(0,\infty) \to L^2(0,\tau)$;

Wiener–Hopf operator: $W(a) := P_+C(a)|L^2(0,\infty)$,
$W_\tau(a) := P_\tau W(a)|L^2(0,\tau)$.

In particular, if $a = \hat{k}$, $k \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ then

$$
\begin{aligned}
(C(\hat{k})f)(x) &= \int_{-\infty}^{\infty} k(x-y) f(y) \,\mathrm{d}y, \\
(W(\hat{k})f)(x) &= \int_{0}^{\infty} k(x-y) f(y) \,\mathrm{d}y, \\
(W_\tau(\hat{k})f)(x) &= \int_{0}^{\tau} k(x-y) f(y) \,\mathrm{d}y.
\end{aligned}
$$

Set

$$
\begin{aligned}
& a_\omega(\xi) := \sqrt{\pi/\omega}\,\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\xi^2/(4\omega)}, \qquad \xi \in \mathbb{R} \\
\Rightarrow \quad & (C(a_\omega)f)(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} f(y) \,\mathrm{d}y.
\end{aligned}
$$

Theorem (Hartman & Wintner) *If $a \in L^\infty(\mathbb{R})$ is real-valued then $\sigma(W(a)) = \operatorname{conv} \mathcal{R}(a)$, where $\mathcal{R}(a)$ is the essential range of a.*

Theorem (Böttcher & Widom) *If $a \in L^\infty(\mathbb{R})$ is real-valued then*

$$\sigma(W_\tau(a)) \subset \sigma(W(a)), \qquad \tau > 0,$$

and $\sigma(W_\tau(a)) \stackrel{\tau\to\infty}{\longrightarrow} \sigma(W(a))$ in the Hausdorff matrix.

Therefore, for real-valued $a \in L^\infty(\mathbb{R})$, $\sigma(W_\tau(a)) \subset \operatorname{conv}\mathcal{R}(a)$, $\tau > 0$, and $\sigma(W_\tau(a)) \stackrel{\tau\to\infty}{\longrightarrow} \mathcal{R}(a)$ in the Hausdorff metric.

Theorem *Unless $\mathcal{R}(a)$ is a singleton, if $a \in L^\infty(\mathbb{R})$ is real-valued then an endpoint of the line segment $\mathcal{R}(a)$ cannot be an eigenvalue of $W_\tau(a)$.*

The Szegő First Limit Theorem *If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is real-valued and $\varphi \in C(\mathbb{R})$ a function s.t. $\varphi(x)/x$ has a finite limit for $x \to 0$ then $\varphi(W_\tau(a))$ is a trace class operator for all $\tau > 0$, $\varphi \circ a \in L^1(\mathbb{R})$ and*

$$\lim_{\tau\to\infty} \frac{\operatorname{tr}\varphi(W_\tau(a))}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(a(\xi))\,\mathrm{d}\xi.$$

Theorem *(A continuous analogue of Widom and Tilli) Let $a \in C(\dot{\mathbb{R}}) \cap L^1(\mathbb{R})$, where $\dot{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$, suppose that $\mathcal{R}(a)$ has no interior points and $\sigma(W(a)) = \mathcal{R}(a)$. Then $\sigma(W_\tau(a)) \to \mathcal{R}(a)$ in the Hausdorff metric and, for $\varphi \in C(\mathbb{C})$ s.t. $\lim_{z\to 0}\varphi(z)/z$ is finite, $\varphi \circ a \in L^1(\mathbb{R})$ and*

$$\lim_{\tau\to\infty} \frac{\operatorname{tr}\varphi(W_\tau(a))}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(a(\xi))\,\mathrm{d}\xi.$$

$\varphi(W_\tau(a))$ being trace class, it is true that

$$\mathrm{tr}\,\varphi(W_\tau(a)) = \sum_{j\in\sigma(W_\tau(a))} \varphi(\lambda_j),$$

where the sum is at most countable.

The function a relevant to Fox–Li is not real-valued. However, real-valuedness can be dropped once we pass from eigenvalues to singular values, i.e. replace $W_\tau(a)$ with $|W_\tau(a)| := (W_\tau(a)W_\tau(a)^*)^{1/2}$:

***Theorem* (Avram & Parter)** *Let $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\varphi \in C[0,\infty)$, s.t. $\lim_{x\to 0} \varphi(x)/x$ is finite. Then $\varphi(|W_\tau(a)|)$ is trace class, $\varphi \circ a \in L^1(\mathbb{R})$ and*

$$\lim_{\tau\to\infty} \frac{\mathrm{tr}\,\varphi(|W_\tau(a)|)}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(|a(\xi)|)\,\mathrm{d}\xi.$$

HIGHLY OSCILLATORY CONVOLUTION-TYPE PROBLEMS

The Fox–Li operator $\mathcal{F}_\omega$ is a convolution operator. Serendipitously, $\mathcal{F}_\omega\mathcal{F}_\omega^*$ is unitarily equivalent to a convolution operator:

***Lemma** Let*

$$V : L^2(-1,1) \to L^2(-1,1), \qquad (Vf)(x) := \mathrm{e}^{-\mathrm{i}\omega x^2} f(x).$$

(Note that V is unitary.) Then

$$(V\mathcal{F}_\omega\mathcal{F}_\omega^* V^* f)(x) = \int_{-1}^{1} \frac{\sin(2\omega(x-y))}{\omega(x-y)} f(y)\,\mathrm{d}y, \qquad x \in (-1,1).$$

The general context:
If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ then $a \in L^2(\mathbb{R})$, hence $\exists k \in L^2(\mathbb{R})$ s.t. $a = \widehat{k}$. Since $\widehat{k} \in L^1(\mathbb{R})$, we know that k is continuous and $k(\pm\infty) = 0$. Set

$$k_\omega(t) := k(\omega t), \qquad t \in \mathbb{R}, \quad \omega > 0$$

and compress the convolution operator $C(\widehat{k}_\omega)$ to $L^2(-1,1)$,

$$(C_{(-1,1)}(\widehat{k}_\omega)f)(x) := \int_{-1}^{1} k(\omega(x-y)) f(y)\,\mathrm{d}y, \qquad x \in (-1,1).$$

We have just proved that $\mathcal{F}_\omega \mathcal{F}_\omega^*$ is unitarily equivalent to

$$C_{(-1,1)}(\widehat{k}_\omega) \qquad \text{with} \qquad k(t) = \frac{\sin(2t)}{t}.$$

Thus,

$$a(\xi) = \widehat{k}(\xi) = \int_{-\infty}^{\infty} \frac{\sin(2t)}{t} \mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t = \pi\chi_{(-2,2)}(\xi),$$

where $\chi_{(\alpha,\beta)}$ is the characteristic function of (α,β).

Let U be the unitary operator s.t.

$$U : L^2(-1,1) \to L^2(0,\tau), \qquad (Uf)(t) := \sqrt{\frac{2}{\tau}} f\left(\frac{2t-\tau}{\tau}\right),$$

Because

$$U^* : L^2(0,\tau) \to L^2(-1,1), \qquad (U^*g)(x) = \sqrt{\frac{\tau}{2}} g\left(\frac{\tau(x+1)}{2}\right),$$

we deduce by direct computation that

$$UC_{(-1,1)}(\widehat{k}_\omega)U^* = \frac{2}{\tau} W_\tau(\widehat{k}_{2\omega/\tau}).$$

Theorem *Provided that* $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *is real-valued, then*

$$\omega\sigma(C_{(-1,1)}(\widehat{k}_\omega)) \subset \operatorname{conv}\mathcal{R}(\widehat{k}), \qquad \omega\sigma(C_{(-1,1)}(\widehat{k}_\omega)) \stackrel{\omega\to\infty}{\longrightarrow} \operatorname{conv}\mathcal{R}(\widehat{k})$$

in the Hausdorff metric. Moreover, if $\varphi \in C(\mathbb{R})$ *and* $\lim_{x\to 0}\varphi(x)/x$ *is finite then*

$$\lim_{\omega\to\infty} \frac{\operatorname{tr}\varphi(\omega C_{(-1,1)}(\widehat{k}_\omega))}{2\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\widehat{k}(\xi))\,\mathrm{d}\xi.$$

The proof follows by letting $\tau = 2\omega$, observing that

$$\omega\sigma(C_{(-1,1)}(\widehat{k}_\omega)) = \sigma(W_{2\omega}(\widehat{k}))$$

and using theorems that we have quoted earlier.

We are now in a position to prove two theorems that describe fairly comprehensively the structure of $s(\mathcal{F}_\omega)$ for the Fox–Li operator.

Theorem A *$s(\mathcal{F}_\omega) \subset [0, \sqrt{\pi/\omega})$, $\omega > 0$.*

Theorem B *As $\omega \to \infty$, the sets $\omega s^2(\mathcal{F}_\omega)$ converge in the Hausdorff metric to the line segment $[0, \pi]$. Moreover, for each $\varepsilon \in (0, \pi/2)$,*

$$|\omega s^2(\mathcal{F}_\omega) \cap (\pi - \varepsilon, \pi)| = \frac{4\omega}{\pi} + o(\omega),$$
$$|\omega s^2(\mathcal{F}_\omega) \cap (\varepsilon, \pi - \varepsilon)| = o(\omega),$$
$$|\omega s^2(\mathcal{F}_\omega) \cap (0, \varepsilon)| = \infty,$$

where $|E|$ is the number of points of E, counting multiplicities.

Proofs of Theorems A & B The operator $\omega \mathcal{F}_\omega \mathcal{F}_\omega^* = \omega C_{(-1,1)}(\widehat{k})$ is unitarily equivalent to

$$\omega \frac{2}{2\omega} W_{2\omega}(\widehat{k}) = W_{2\omega}(\pi \chi_{(-2,2)}).$$

Hence $\omega s^2(\mathcal{F}_\omega) \subset \operatorname{conv} \mathcal{R}(\pi \chi_{(-2,2)}) = [0, \pi]$ for all $\omega > 0$ and converges to $[0, \pi]$ in the Hausdorff metric for $\omega \to \infty$. Moreover $\pi \notin \omega s^2(\mathcal{F}_\omega)$, otherwise it would be an eigenvalue of $W_{2\omega}(\pi \chi_{(-2,2)})$, a contradiction. This proves Theorem A and first part of Theorem B.

To complete the proof, let $0 < \alpha < \beta \le \pi$ and choose

$$\varphi, \psi \in C(\mathbb{R}) \quad \text{s.t.} \quad \varphi(x) = \psi(x) = 0, \quad -\infty < x < \alpha/2,$$

and $\varphi(x) \le \chi_{(\alpha,\beta)} \le \psi(x)$, $x \in [0, \pi]$. Let $N_{(\alpha,\beta)} := |\omega s^2(\mathcal{F}_\omega) \cap (\alpha, \beta)|$, then

$$N_{(\alpha,\beta)} = \operatorname{tr} \chi_{(\alpha,\beta)}(W_{2\omega}(\pi\chi_{(-2,2)})).$$

Since $\chi_{(\alpha,\beta)} \le \psi$,

$$\limsup_{\omega\to\infty} \frac{N_{(\alpha,\beta)}}{2\omega} \le \lim_{\omega\to\infty} \frac{\operatorname{tr} \psi(W_{2\omega}(\pi\chi_{(-2,2)}))}{2\omega}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\pi\chi_{(-2,2)}(\xi))\, d\xi = \frac{1}{2\pi}\int_{-2}^{2} \psi(\pi)\, d\xi = \frac{2}{\pi}\psi(\pi),$$

likewise

$$\liminf_{\omega\to\infty} \frac{N_{(\alpha,\beta)}}{2\omega} \ge \frac{2}{\pi}\varphi(\pi).$$

For $(\alpha, \beta) = (\pi - \varepsilon, \pi)$ we choose φ, ψ s.t. $\varphi(\pi) = \psi(\pi) = 1$, therefore

$$\lim_{\omega\to\infty} \frac{N_{(\pi-\varepsilon,\pi)}}{2\omega} = \frac{2}{\pi} \qquad \Rightarrow \qquad N_{(\pi-\varepsilon,\pi)} = \frac{4\omega}{\pi} + o(\omega).$$

Likewise, for $(\alpha, \beta) = (\varepsilon, \pi - \varepsilon)$ we take $\varphi(\pi) = \psi(\pi) = 0$ and deduce that $N_{(\varepsilon,\pi-\varepsilon)} = o(\omega)$.

Finally, since the endpoint of $\operatorname{conv}\mathcal{R}(a)$ isn't an eigenvalue, $W_{2\omega}(\pi\chi_{(-2,2)})$ is injective, hence so is $\omega\mathcal{F}_\omega\mathcal{F}_\omega^*$. Therefore $\omega\mathcal{F}_\omega\mathcal{F}_\omega^*$ has dense, thus infinite-dimensional range. We deduce that $\omega s^2(\mathcal{F}_\omega) \cap (0, \pi)$ is an infinite set. But only $4\omega/\pi + o(\omega)$ of its points live in $[\varepsilon, \pi)$ and so infinitely many do so in $(0, \varepsilon)$. $\square$

Note that, for any $\alpha < \beta$, $0 \notin [\alpha, \beta]$, one obtains similarly that

$$\operatorname{mes}\left[\{\xi \in \mathbb{R} : \widehat{k}(\xi) = \alpha\} \cup \{\xi \in \mathbb{R} : \widehat{k}(\xi) = \beta\}\right] = 0$$

$$\Rightarrow \quad \lim_{\omega\to\infty} \frac{|\sigma(\omega C_{(-1,1)}(\widehat{k}_\omega)) \cap (\alpha, \beta)|}{2\omega} = \frac{1}{2\pi}\operatorname{mes}\{\xi \in \mathbb{R} : \widehat{k}(\xi) \in (\alpha, \beta)\}.$$

Remarkably, we have all these results *because* of high oscillation. It is an oft-repeated lesson: once you understand high oscillation mathematically, it is not a barrier to understanding, it is a friend!

EXAMPLES

- $k(t) = \mathrm{e}^{-t^2}$

Now $a(\xi) = \widehat{k}(\xi) = \sqrt{\pi}\mathrm{e}^{-\xi^2/4}$, hence $\omega C_{(-1,1)}(\widehat{k}) \subset [0, \sqrt{\pi})$, fills it densely for $\omega \to \infty$ and the number of eigenvalues of $C_{(-1,1)}(\widehat{k})$ in $(\alpha/\omega, \beta/\omega)$ is

$$\frac{\omega}{\pi}\mathrm{mes}\,\{\xi \in \mathbb{R} \,:\, \alpha < \sqrt{\pi}\mathrm{e}^{-\xi^2/4} < \beta\} + o(\omega).$$

- $k(t) = \dfrac{1 - \cos t}{t^2}$

Now $a(\xi) = \widehat{k}(\xi) = \pi(1 - |\xi|)_+$, $\sigma(\omega C_{(-1,1)}(\widehat{k}_\omega))$ fills $[0, \pi]$ densely and, for $0 < \alpha < \beta \leq \pi$, the number of eigenvalues of $C_{(-1,1)}(\widehat{k}_\omega)$ in $(\alpha/\omega, \beta/\omega)$ is

$$\frac{\omega}{\pi}\mathrm{mes}\,\{\xi \,:\, \alpha < \pi(1 - |\xi|)_+ < \beta\} + o(\omega) = \frac{2(\beta - \alpha)}{\pi^2}\omega + o(\omega).$$

ON THE IMPORTANCE OF BEING $L^1(\mathbb{R})$

The Fox–Li kernel $k(t) = \mathrm{e}^{\mathrm{i}t^2}$ is not $L^1(\mathbb{R})$, and this is the source of our problems! Once $k \in L^1(\mathbb{R})$, life becomes simple:

Theorem *If $k \in L^1(\mathbb{R})$, $k(t) = k(-t)$ and $\mathcal{R}(\widehat{k})$ has no interior points, then $\omega\sigma(C_{(-1,1)}(\widehat{k}_\omega))$ converges to $\mathcal{R}(\widehat{k})$ in the Hausdorff metric and, for every $\varphi \in C(\mathbb{C})$ such that $\lim_{z\to 0}\varphi(z)/z$ is finite, $\varphi \circ a \in L^1(\mathbb{R})$ and*

$$\lim_{\omega\to\infty}\frac{1}{2\omega}\sum_{\lambda\in\sigma(C_{(-1,1)}(\widehat{k}_\omega))}\varphi(\omega\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\widehat{k}(\xi))\,\mathrm{d}\xi.$$

Theorem *Let $k \in L^1(\mathbb{R})$. Then $\omega s(C_{(-1,1)}(\widehat{k}_\omega)) \subset \mathcal{R}(|\widehat{k}|)$ and converges to it in the Hausdorff metric as $\omega \to \infty$. If $\varphi \in C([0,\infty))$ and $\lim_{x\to 0}\varphi(x)/x$ is finite then $\varphi \circ \widehat{k} \in L^1(\mathbb{R})$ and*

$$\lim_{\omega\to\infty}\frac{1}{2\omega}\sum_{\lambda\in s(C_{(-1,1)}(\widehat{k}_\omega))}\varphi(\omega\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(|\widehat{k}|)\,\mathrm{d}\xi.$$

EXAMPLE: Attenuated Fox–Li.

Let $k(t) = \mathrm{e}^{(\mathrm{i}-\varepsilon)t^2}$ for $\varepsilon > 0$:

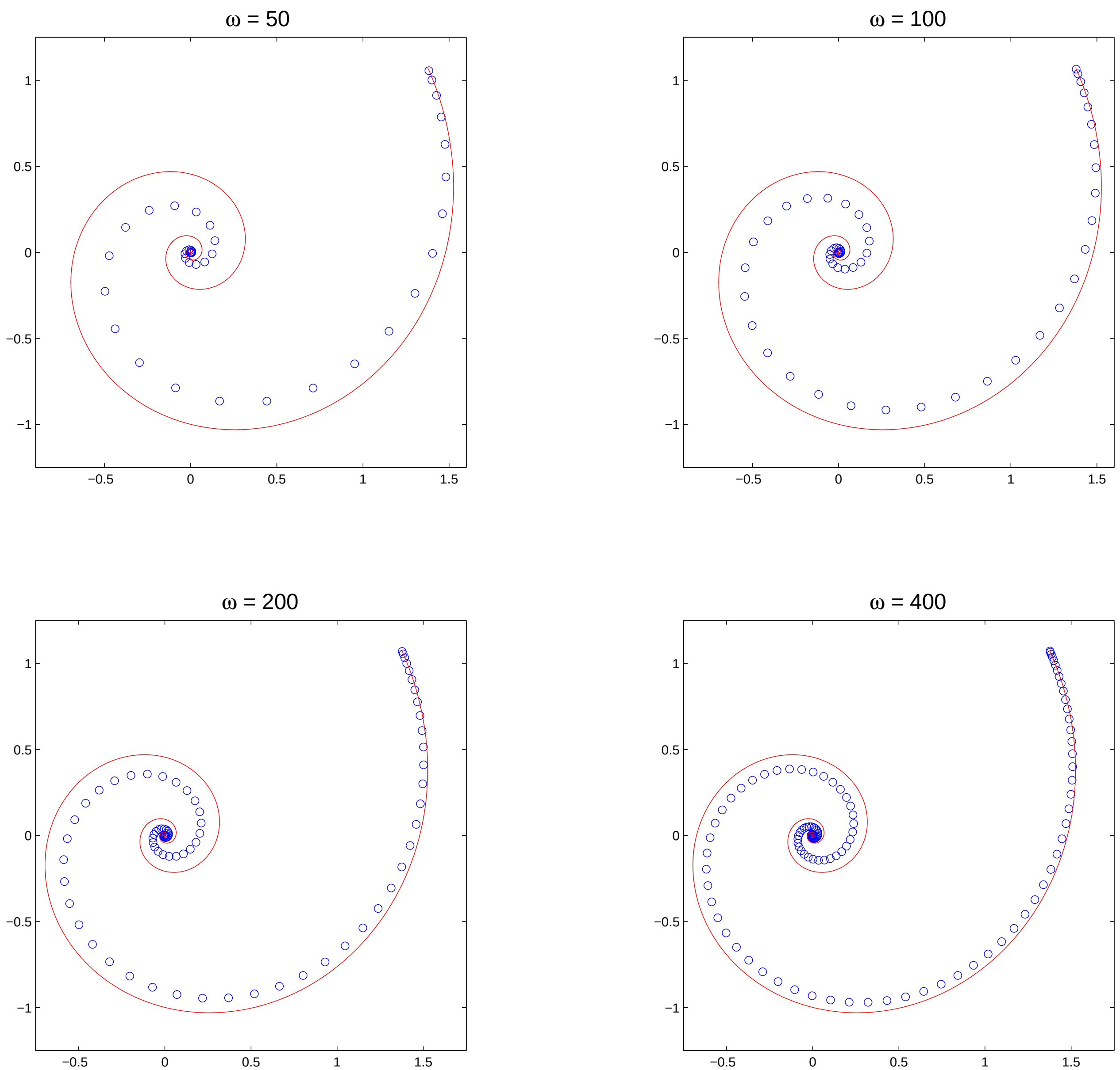
$$(\mathcal{F}_{\omega,\varepsilon} f)(x) := \int_{-1}^{1} \mathrm{e}^{(\mathrm{i}-\varepsilon)\omega(x-y)^2} f(y)\,\mathrm{d}y, \qquad x \in (-1,1).$$

Since

$$\widehat{k}(\xi) = \sqrt{\frac{\pi(\varepsilon+\mathrm{i})}{1+\varepsilon^2}}\exp\left(-\frac{\varepsilon\xi^2}{4(1+\varepsilon^2)}\right)\exp\left(-\mathrm{i}\frac{\xi^2}{4(1+\varepsilon^2)}\right),$$

$\mathcal{R}(\widehat{k})$ is a spiral, rotating clockwise from $\frac{\pi(\varepsilon+\mathrm{i})}{1+\varepsilon^2}$ to the origin. Thus, for $\omega \to \infty$, $\sqrt{\omega}\sigma(\mathcal{F}_{\omega,\varepsilon})$ converges to the spiral

$$\left\{\frac{\pi(\varepsilon+\mathrm{i})}{1+\varepsilon^2}\mathrm{e}^{-(\mathrm{i}+\varepsilon)t} \,:\, t \geq 0\right\}.$$



Spectra of $\mathcal{F}_{\omega,\varepsilon}$ for $\varepsilon = \frac{1}{4}$ and different values of ω, as well as the spiral $\widehat{k}$.

SPECULATING ON THE FOX–LI SPECTRUM

ATTEMPT 1: Wiener–Hopf operators

Although $\mathcal{F}_\omega$ is unitarily equivalent to

$$\frac{1}{\sqrt{\omega}} W_{2\sqrt{\omega}}(a) \qquad \text{with} \qquad a(\xi) = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\xi^2/4},$$

our theory is inapplicable because $a \notin C(\dot{\mathbb{R}}) \cap L^1(\mathbb{R})$. Consider instead

$$\ell^{[\omega]}(t) := \chi_{(-2\sqrt{\omega},2\sqrt{\omega})}(t)\mathrm{e}^{\mathrm{i}t^2} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Since $\ell^{[\omega]}_{\sqrt{\omega}}(t) := \ell^{[\omega]}(\sqrt{\omega}t) = \chi_{(-2,2)}(t)\mathrm{e}^{\mathrm{i}\omega t^2}$, we deduce that

$$\mathcal{F}_\omega = C_{(-1,1)}\left(\ell^{[\omega]}_{\sqrt{\omega}}\right).$$

From this it is possible to deduce that

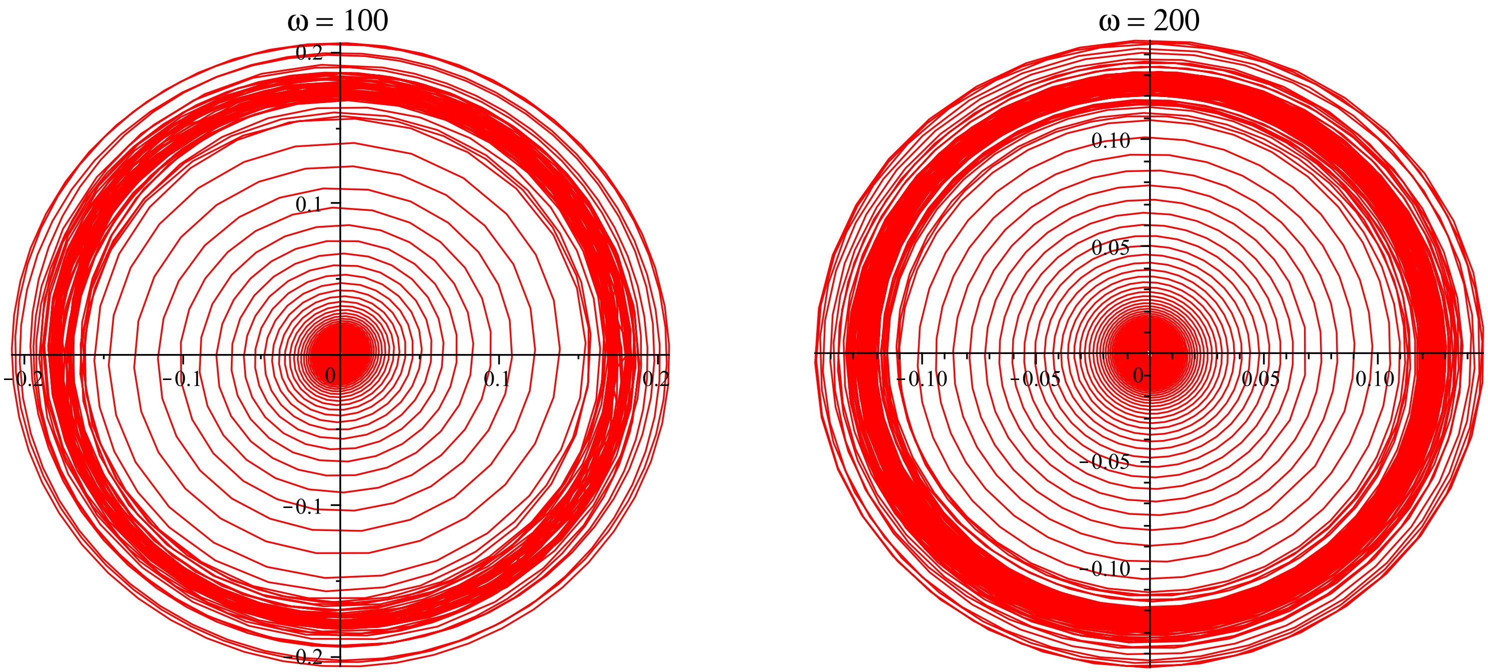
$$\sqrt{\omega}\sigma(\mathcal{F}_\omega) = \sigma(W_{2\sqrt{\omega}}(\widehat{\ell}^{[\omega]})).$$

The problem is that the right-hand side depends upon ω in two different ways.

It is possible to show that $\sigma(W_\tau(\hat{\ell}^{[\omega]}))$ is asymptotically distributed along $\mathcal{R}(\hat{\ell}^{[\omega]})$ for $\omega \to \infty$. The snag is that our $\tau = 2\sqrt{\omega}$ depends upon ω.

Let us assume (wrongly!) that for $\omega \gg 1$ we can replace convolution over $(0, 2\sqrt{\omega})$ by one over $(0, \infty)$. This leads to the 'conclusion' that

$$\sigma(\mathcal{F}_\omega) \approx (1/\sqrt{\omega})\mathcal{R}(\hat{\ell}^{[\omega]}).$$



The spirals $(1/\sqrt{\omega})\mathcal{R}(\hat{\ell}^{[\omega]})$.

ATTEMPT 2: Toeplitz operators

Fix $\omega > 0$ and discretize $\mathcal{F}_\omega$ at $2N+1$ equidistant points in $[-1,1]$. This approximates the spectral problem by the algebraic eigenvalue problem

$$B^{[N]}\boldsymbol{f}^{[N]} = \lambda^{[N]}\boldsymbol{f}^{[N]},$$

where

$$B^{[N]} := (v_{j-k}^{[N]})_{j,k=-N}^{N} \quad \text{with} \quad v_n^{[N]} := \frac{\mathrm{e}^{\mathrm{i}\omega n^2/N^2}}{N} \quad \text{is Toepliz.}$$

Given $v \in L^1(\mathbb{T})$ with the Fourier coefficients

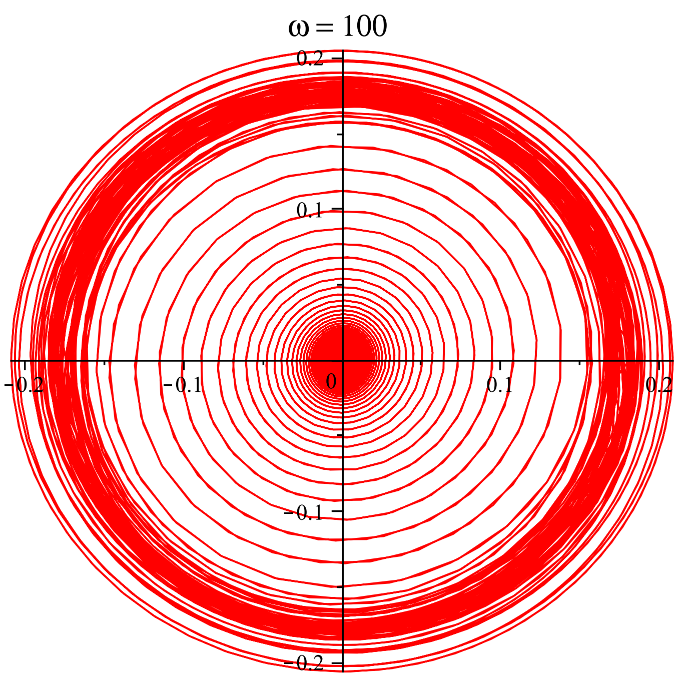
$$v_n := \frac{1}{2\pi}\int_0^{2\pi} v(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta, \qquad n \in \mathbb{Z},$$

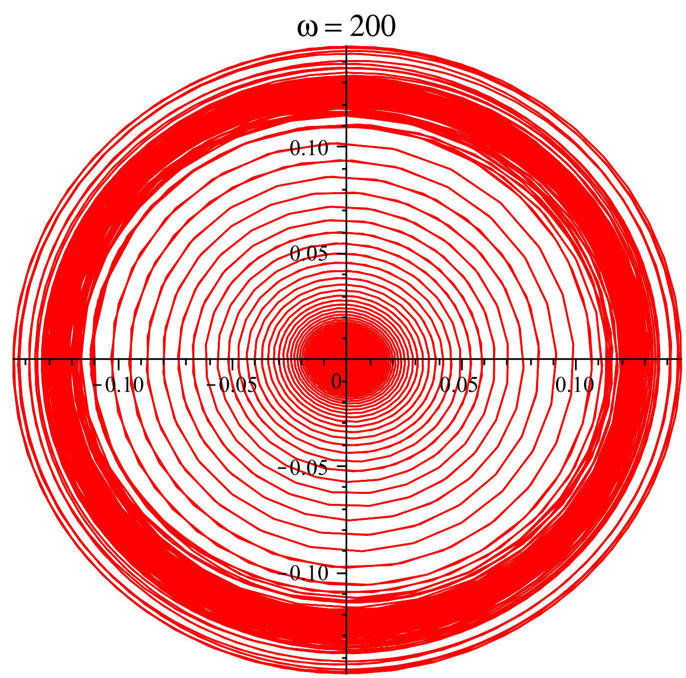
we denote by $T(v)$ and $T_N(v)$ the infinite Toeplitz matrix $(v_{j-k})_{j,k\in\mathbb{Z}_+}$ and finite matrix $B^{[N]} = T_N(v^{[N]})$ resp., where in our case

$$v^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=-2N}^{2N} v_n^{[N]}\mathrm{e}^{\mathrm{i}n\theta} = \frac{1}{N}\sum_{n=-2N}^{2N} \mathrm{e}^{\mathrm{i}\omega n^2/N^2}\mathrm{e}^{\mathrm{i}n\theta}.$$

Since Fox–Li is compact, $\sigma(T_N(v^{[N]})) \to \sigma(\mathcal{F}_\omega)$, but since *both* the dimension *and* the generating function vary with N, we have no theoretical tools to predict the limit of $\sigma(T_N(v^{[N]}))$. We may again abandon rigour and replace

$$\sigma(T_N(v^{[N]})) \approx \sigma(T(v^{[N]})) = v^{[N]}(\mathbb{T}) \qquad \Rightarrow \qquad \sigma(\mathcal{F}_\omega) \approx v^{[N]}(\mathbb{T}).$$





The spirals $v^{[N]}(\mathbb{T})$ for $N = 500$.

The Wiener–Hopf and Toeplitz attempts are equally wrong:

$$v^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{N}\sum_{n=-2N}^{2N} \mathrm{e}^{\mathrm{i}\omega n^2/N^2}\mathrm{e}^{\mathrm{i}\sqrt{\omega}\xi n/N} = \int_{-2}^{2} \mathrm{e}^{\mathrm{i}\omega x^2}\mathrm{e}^{\mathrm{i}\sqrt{\omega}\xi x}\,\mathrm{d}x + O(1/N)$$

$$= \frac{1}{\sqrt{\omega}}\int_{-2\sqrt{\omega}}^{2\sqrt{\omega}} \mathrm{e}^{\mathrm{i}t^2}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t + O(1/N) = \frac{1}{\sqrt{\omega}}\widehat{\ell}^{[\omega]}(\xi) + O(1/N).$$

Moreover, using asymptotic analysis,

$$\widehat{\ell}^{[\omega]}(\xi) = \frac{\sqrt{\pi}\mathrm{e}^{-\mathrm{i}\xi^2/4}}{2\sqrt{-\mathrm{i}\omega}}\left[\mathrm{erf}\left(2\sqrt{-\mathrm{i}\omega} + \tfrac{1}{2}\sqrt{-\mathrm{i}}\xi\right) + \mathrm{erf}\left(2\sqrt{-\mathrm{i}\omega} - \tfrac{1}{2}\sqrt{-\mathrm{i}}\xi\right)\right]$$

therefore

$$4\sqrt{\omega} > |\xi|: \quad \widehat{\ell}^{[\omega]}(\xi) \approx \frac{\sqrt{\mathrm{i}\pi}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\xi^2}}{\sqrt{\omega}} - \frac{\mathrm{i}\mathrm{e}^{4\mathrm{i}\omega}}{\sqrt{\omega}}\left(\frac{\mathrm{e}^{-2\mathrm{i}\sqrt{\omega}\xi}}{4\sqrt{\omega}-\xi} + \frac{\mathrm{e}^{2\mathrm{i}\sqrt{\omega}\xi}}{4\sqrt{\omega}+\xi}\right) + O(\omega^{-2});$$

$$4\sqrt{\omega} < |\xi|: \quad \widehat{\ell}^{[\omega]}(\xi) \approx \frac{\mathrm{i}\mathrm{e}^{4\mathrm{i}\omega}}{\sqrt{\omega}}\left(\frac{\mathrm{e}^{-2\mathrm{i}\sqrt{\omega}\xi}}{\xi - 4\sqrt{\omega}} - \frac{\mathrm{e}^{2\mathrm{i}\sqrt{\omega}\xi}}{\xi + 4\sqrt{\omega}}\right) + O(\xi^{-2}).$$

This explains the two regimes visible in the figures: extended rotation with roughly equal amplitude, followed by attenuation.

ATTEMPT 3: Theta-three

Compare

$$v^{[N]}(\mathrm{e}^{2\mathrm{i}\alpha}) = \frac{1}{N}\left[1 + 2\sum_{k=1}^{2N} q_n^{k^2}\cos(2\alpha k)\right], \qquad q_N = \mathrm{e}^{\mathrm{i}\omega/N^2}, \quad |q_N| = 1,$$

$$\theta_3(\alpha|q) := 1 + 2\sum_{k=1}^{\infty} q^{k^2}\cos(2\alpha k), \qquad q \in \mathbb{C}, \quad |q| < 1.$$

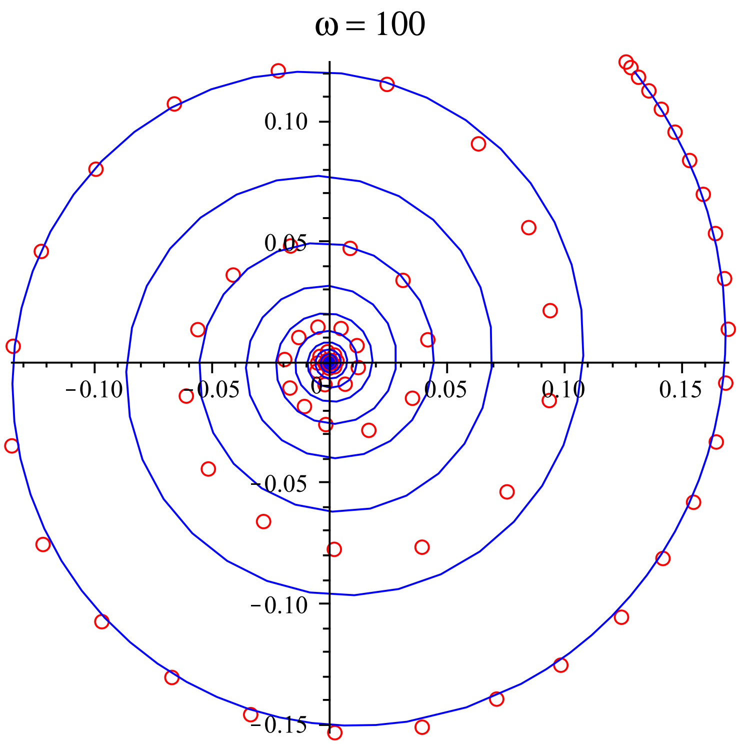
What makes $v^{[N]}$ stay nice, in spite of $|q_N| = 1$, is the presence of the $1/N$ factor.

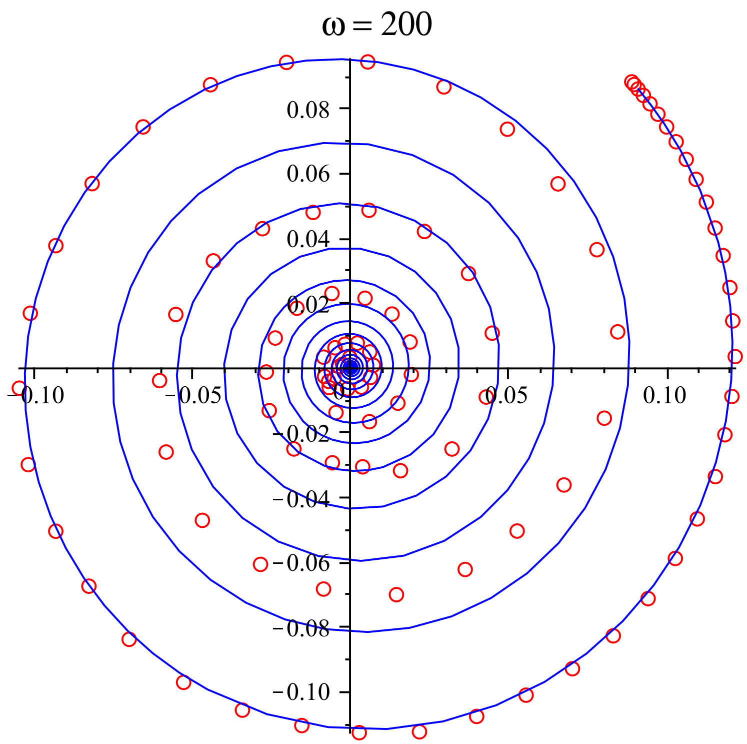
θ_3 blows up for $|q| = 1$. Instead, let's take

$$q_{N,\omega} = \left(1 - \frac{\sqrt{\omega}}{\sqrt{2}N^2}\right)q_N, \qquad 1 > |q_{N,\omega}| = 1 + O(1/N^2)$$

and plot

$$\frac{\theta_3(\alpha|q_{N,\omega})}{N} \qquad \text{for} \qquad N \gg 1, \quad \alpha \in [-\pi/2, \pi/2].$$





Attenuated theta spirals, superimposed on the spectra.

What is the explanation of this remarkable fit, at least near the ‘head’ of the spiral? We have no idea.