On the singular values and eigenvalues of the Fox–Li and related operators

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THE FOX–LI OPERATOR

\[(\mathcal{F}_\omega f)(x) := \int_{-1}^{1} e^{i\omega(x-y)^2} f(y) \, dy, \quad x \in (-1, 1),\]

where \(\omega > 0\):

- \(\mathcal{F}_\omega\) is a complex-valued, symmetric linear operator in \(L^2(-1, 1)\). It is compact, therefore \(\sigma(\mathcal{F}_\omega)\) consists of the origin and at most a countable number of eigenvalues accumulating at the origin.

- Its spectrum \(\sigma(\mathcal{F}_\omega)\) is important in laser and maser engineering: the eigenfunctions represent modes (self-reproducing field distributions) between two semi-circular reflectors placed at a fixed distance from each other.

- Computational results indicate that \(\sigma(\mathcal{F}_\omega)\) lies on a spiral commencing near \(\sqrt{\pi/\omega} e^{i\pi/4}\) and rotating clockwise to the origin. Not much is known of the precise shape of this spiral.
Fox–Li eigenvalues for $\omega = 100$ and $\omega = 200$. 

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FOX–LI SINGULAR VALUES

Singular values $s(F_\omega)$ are the positive square roots of the eigenvalues of the positive semidefinite operator $F_\omega^* F$. They are of an independent interest, e.g. in random matrix theory.

Fox–Li singular values, approximated as eigenvalues of a $(2N + 1) \times (2N + 1)$ matrix.

We observe that almost all singular values accumulate at just two points: the origin and something which suspiciously looks like $\sqrt{\pi/\omega}$. 
WIENER–HOPF OPERATORS

The Fourier–Plancharel transform \( F : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \):

\[
\hat{f}(\xi) = (Ff)(\xi) := \int_{-\infty}^{\infty} f(t) e^{i\xi t} \, dt, \quad \xi \in \mathbb{R}.
\]

Few definitions:

- **Multiplication:** \( M(a) : f \mapsto af \);
- **Convolution:** \( C(a) : f \mapsto F^{-1}M(a)Ff \);
- **Projection:** \( P_+ \) orthogonally projects \( L^2(\mathbb{R}) \to L^2(0, \infty) \), \( P_\tau \) orthogonally projects \( L^2(0, \infty) \to L^2(0, \tau) \);
- **Wiener–Hopf operator:** \( W(a) := P_+ C(a)|L^2(0, \infty) \), \( W_\tau(a) := P_\tau W(a)|L^2(0, \tau) \).
In particular, if $a = \hat{k}$, $k \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ then

$$
(C(\hat{k})f)(x) = \int_{-\infty}^{\infty} k(x - y) f(y) \, dy,
$$

$$
(W(\hat{k})f)(x) = \int_{0}^{\infty} k(x - y) f(y) \, dy,
$$

$$
(W_\tau(\hat{k})f)(x) = \int_{0}^{\tau} k(x - y) f(y) \, dy.
$$

Set

$$
a_\omega(\xi) := \sqrt{\pi/\omega} e^{i\pi/4} e^{-i\xi^2/(4\omega)}, \quad \xi \in \mathbb{R}
$$

$$
\Rightarrow (C(a_\omega)f)(x) = \int_{-\infty}^{\infty} e^{i\omega(x-y)^2} f(y) \, dy.
$$

**Theorem (Hartman & Wintner)** If $a \in L^\infty(\mathbb{R})$ is real-valued then

$$
\sigma(W(a)) = \text{conv} \, \mathcal{R}(a), \quad \text{where } \mathcal{R}(a) \text{ is the essential range of } a.
$$

**Theorem (Böttcher & Widom)** If $a \in L^\infty(\mathbb{R})$ is real-valued then

$$
\sigma(W_\tau(a)) \subset \sigma(W(a)), \quad \tau > 0,
$$

and $\sigma(W_\tau(a)) \xrightarrow{\tau \to \infty} \sigma(W(a))$ in the Hausdorff matrix.
Therefore, for real-valued $a \in L^\infty(\mathbb{R})$, $\sigma(W_\tau(a)) \subset \text{conv } \mathcal{R}(a)$, $\tau > 0$, and $\sigma(W_\tau(a)) \xrightarrow{\tau \to \infty} \mathcal{R}(a)$ in the Hausdorff metric.

**Theorem**  Unless $\mathcal{R}(a)$ is a singleton, if $a \in L^\infty(\mathbb{R})$ is real-valued then an endpoint of the line segment $\mathcal{R}(a)$ cannot be an eigenvalue of $W_\tau(a)$.

**The Szegő First Limit Theorem** If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is real-valued and $\varphi \in C(\mathbb{R})$ a function s.t. $\varphi(x)/x$ has a finite limit for $x \to 0$ then $\varphi(W_\tau(a))$ is a trace class operator for all $\tau > 0$, $\varphi \circ a \in L^1(\mathbb{R})$ and

$$\lim_{\tau \to \infty} \frac{\text{tr } \varphi(W_\tau(a))}{\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(a(\xi)) \, d\xi.$$ 

**Theorem** (A continuous analogue of Widom and Tilli) Let $a \in C(\hat{\mathbb{R}}) \cap L^1(\mathbb{R})$, where $\hat{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$, suppose that $\mathcal{R}(a)$ has no interior points and $\sigma(W(a)) = \mathcal{R}(a)$. Then $\sigma(W_\tau(a)) \to \mathcal{R}(a)$ in the Hausdorff metric and, for $\varphi \in C(\mathbb{C})$ s.t. $\lim_{z \to 0} \varphi(z)/z$ is finite, $\varphi \circ a \in L^1(\mathbb{R})$ and

$$\lim_{\tau \to \infty} \frac{\text{tr } \varphi(W_\tau(a))}{\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(a(\xi)) \, d\xi.$$ 


\(\varphi(W_\tau(a))\) being trace class, it is true that
\[
\text{tr } \varphi(W_\tau(a)) = \sum_{j \in \sigma(W_\tau(a))} \varphi(\lambda_j),
\]
where the sum is at most countable.

The function \(a\) relevant to Fox–Li is not real-valued. However, real-valuedness can be dropped once we pass from eigenvalues to singular values, i.e. replace \(W_\tau(a)\) with \(|W_\tau(a)| := (W_\tau(a)W_\tau(a)^*)^{1/2}\):

**Theorem (Avram & Parter)** Let \(a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})\) and \(\varphi \in C[0, \infty), \text{ s.t. } \lim_{x \to 0} \varphi(x)/x \text{ is finite. Then } \varphi(|W_\tau(a)|) \text{ is trace class, } \varphi \circ a \in L^1(\mathbb{R})\) and
\[
\lim_{\tau \to \infty} \frac{\text{tr } \varphi(|W_\tau(a)|)}{\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(|a(\xi)|) \, d\xi.
\]
HIGHLY OSCILLATORY CONVOLUTION-TYPE PROBLEMS

The Fox–Li operator $\mathcal{F}_\omega$ is a convolution operator. Serendipitously, $\mathcal{F}_\omega \mathcal{F}_\omega^*$ is unitarily equivalent to a convolution operator:

**Lemma** Let

$$V : L^2(-1, 1) \to L^2(-1, 1), \quad (V f)(x) := e^{-i\omega x^2} f(x).$$

(Note that $V$ is unitary.) Then

$$(V \mathcal{F}_\omega \mathcal{F}_\omega^* V^* f)(x) = \int_{-1}^{1} \frac{\sin(2\omega(x - y))}{\omega(x - y)} f(y) \, dy, \quad x \in (-1, 1).$$

The general context:

If $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ then $a \in L^2(\mathbb{R})$, hence $\exists k \in L^2(\mathbb{R})$ s.t. $a = \hat{k}$. Since $\hat{k} \in L^1(\mathbb{R})$, we know that $k$ is continuous and $k(\pm \infty) = 0$. Set

$$k_\omega(t) := k(\omega t), \quad t \in \mathbb{R}, \quad \omega > 0$$

and compress the convolution operator $C(\hat{k}_\omega)$ to $L^2(-1, 1)$,

$$(C_{(-1,1)}(\hat{k}_\omega) f)(x) := \int_{-1}^{1} k(\omega(x - y)) f(y) \, dy, \quad x \in (-1, 1).$$
We have just proved that $\mathcal{F}_\omega \mathcal{F}_\omega^*$ is unitarily equivalent to

$$C'_{(-1,1)}(\hat{k}_\omega) \quad \text{with} \quad k(t) = \frac{\sin(2t)}{t}.$$ 

Thus,

$$a(\xi) = \hat{k}(\xi) = \int_{-\infty}^{\infty} \frac{\sin(2t)}{t} e^{i\xi t} \, dt = \pi \chi_{(-2,2)}(\xi),$$

where $\chi_{(\alpha,\beta)}$ is the characteristic function of $(\alpha, \beta)$.

Let $U$ be the unitary operator s.t.

$$U : L^2(-1, 1) \to L^2(0, \tau), \quad (Uf)(t) := \sqrt{\frac{2}{\tau}} f\left(2t - \frac{\tau}{2}\right),$$

Because

$$U^* : L^2(0, \tau) \to L^2(-1, 1), \quad (U^*g)(x) = \sqrt{\frac{\tau}{2}} g\left(\frac{\tau(x + 1)}{2}\right),$$

we deduce by direct computation that

$$UC'_{(-1,1)}(\hat{k}_\omega)U^* = 2 \frac{\tau}{T} W_{\tau}(\hat{k}_{2\omega/\tau}).$$
**Theorem** Provided that $a \in L^{\infty}(\mathbb{R}) \cap L^{1}(\mathbb{R})$ is real-valued, then

$$
\omega \sigma(C_{(-1,1)}(\hat{k} \omega)) \subset \text{conv } R(\hat{k}), \quad \omega \sigma(C_{(-1,1)}(\hat{k} \omega)) \xrightarrow{\omega \to \infty} \text{conv } R(\hat{k})
$$
in the Hausdorff metric. Moreover, if $\varphi \in C(\mathbb{R})$ and $\lim_{x \to 0} \varphi(x)/x$ is finite then

$$
\lim_{\omega \to \infty} \frac{\text{tr } \varphi(\omega C_{(-1,1)}(\hat{k} \omega))}{2\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\hat{k}(\xi)) \, d\xi.
$$

The proof follows by letting $\tau = 2\omega$, observing that

$$
\omega \sigma(C_{(-1,1)}(\hat{k} \omega)) = \sigma(W_{2\omega}(\hat{k}))
$$

and using theorems that we have quoted earlier.

We are now in a position to prove two theorems that describe fairly comprehensively the structure of $s(F\omega)$ for the Fox–Li operator.
**Theorem A** $s(F_\omega) \subset [0, \sqrt{\pi/\omega}), \omega > 0$.

**Theorem B** As $\omega \to \infty$, the sets $\omega s^2(F_\omega)$ converge in the Hausdorff metric to the line segment $[0, \pi]$. Moreover, for each $\varepsilon \in (0, \pi/2)$,

\[
|\omega s^2(F_\omega) \cap (\pi - \varepsilon, \pi)| = \frac{4\omega}{\pi} + o(\omega), \\
|\omega s^2(F_\omega) \cap (\varepsilon, \pi - \varepsilon)| = o(\omega), \\
|\omega s^2(F_\omega) \cap (0, \varepsilon)| = \infty,
\]

where $|E|$ is the number of points of $E$, counting multiplicities.

**Proofs of Theorems A & B** The operator $\omega F_\omega F_\omega^* = \omega C_{(-1,1)}(\hat{k})$ is unitarily equivalent to

\[
\omega \frac{2}{2\omega} W_{2\omega}(\hat{k}) = W_{2\omega}(\pi \chi_{(-2,2)}).
\]

Hence $\omega s^2(F_\omega) \subset \text{conv} \mathcal{R}(\pi \chi_{(-2,2)}) = [0, \pi]$ for all $\omega > 0$ and converges to $[0, \pi]$ in the Hausdorff metric for $\omega \to \infty$. Moreover $\pi \notin \omega s^2(F_\omega)$, otherwise it would be an eigenvalue of $W_{2\omega}(\pi \chi_{(-2,2)})$, a contradiction. This proves Theorem A and first part of Theorem B.
To complete the proof, let $0 < \alpha < \beta \leq \pi$ and choose

$$\varphi, \psi \in C(\mathbb{R}) \quad \text{s.t.} \quad \varphi(x) = \psi(x) = 0, \quad -\infty < x < \alpha/2,$$

and $\varphi(x) \leq \chi(\alpha, \beta) \leq \psi(x), \ x \in [0, \pi]$. Let $N(\alpha, \beta) := |\omega s^2(\mathcal{F} \omega) \cap (\alpha, \beta)|$, then

$$N(\alpha, \beta) = \text{tr} \chi(\alpha, \beta)(W_{2\omega}(\pi \chi(-2,2))).$$

Since $\chi(\alpha, \beta) \leq \psi$,

$$\limsup_{\omega \to \infty} \frac{N(\alpha, \beta)}{2\omega} \leq \lim_{\omega \to \infty} \frac{\text{tr} \psi(W_{2\omega}(\pi \chi(-2,2)))}{2\omega}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\pi \chi(-2,2)(\xi)) \, d\xi = \frac{1}{2\pi} \int_{-2}^{2} \psi(\pi) \, d\xi = \frac{2}{\pi} \psi(\pi),$$

likewise

$$\liminf_{\omega \to \infty} \frac{N(\alpha, \beta)}{2\omega} \geq \frac{2}{\pi} \varphi(\pi).$$

For $(\alpha, \beta) = (\pi - \varepsilon, \pi)$ we choose $\varphi, \psi$ s.t. $\varphi(\pi) = \psi(\pi) = 1$, therefore

$$\lim_{\omega \to \infty} \frac{N(\pi - \varepsilon, \pi)}{2\omega} = \frac{2}{\pi} \quad \Rightarrow \quad N(\pi - \varepsilon, \pi) = \frac{4\omega}{\pi} + o(\omega).$$
Likewise, for \((\alpha, \beta) = (\varepsilon, \pi - \varepsilon)\) we take \(\varphi(\pi) = \psi(\pi) = 0\) and deduce that \(N(\varepsilon, \pi - \varepsilon) = o(\omega)\).

Finally, since the endpoint of \(\text{conv} \mathcal{R}(a)\) isn’t an eigenvalue, \(W_{2\omega}(\pi \chi_{(-2,2)})\) is injective, hence so is \(\omega \mathcal{F}_\omega \mathcal{F}_\omega^*\). Therefore \(\omega \mathcal{F}_\omega \mathcal{F}_\omega^*\) has dense, thus infinite-dimensional range. We deduce that \(\omega s^2(\mathcal{F}_\omega) \cap (0, \pi)\) is an infinite set. But only \(4\omega/\pi + o(\omega)\) of its points live in \([\varepsilon, \pi)\) and so infinitely many do so in \((0, \varepsilon)\). \(\square\)

Note that, for any \(\alpha < \beta, 0 \notin [\alpha, \beta]\), one obtains similarly that

\[
\text{mes} \left[ \{\xi \in \mathbb{R} : \hat{k}(\xi) = \alpha\} \cup \{\xi \in \mathbb{R} : \hat{k}(\xi) = \beta\} \right] = 0
\]

\[
\Rightarrow \lim_{\omega \to \infty} \frac{|\sigma(\omega C_{(-1,1)}(\hat{k}_\omega)) \cap (\alpha, \beta)|}{2\omega} = \frac{1}{2\pi} \text{mes} \{\xi \in \mathbb{R} : \hat{k}(\xi) \in (\alpha, \beta)\}.
\]

Remarkably, we have all these results because of high oscillation. It is an oft-repeated lesson: once you understand high oscillation mathematically, it is not a barrier to understanding, it is a friend!
EXAMPLES

- $k(t) = e^{-t^2}$

Now $a(\xi) = \hat{k}(\xi) = \sqrt{\pi}e^{-\xi^2/4}$, hence $\omega C'_{(-1,1)}(\hat{k}) \subset [0, \sqrt{\pi})$, fills it densely for $\omega \to \infty$ and the number of eigenvalues of $C'_{(-1,1)}(\hat{k})$ in $(\alpha/\omega, \beta/\omega)$ is

$$\frac{\omega}{\pi} \text{mes} \{ \xi \in \mathbb{R} : \alpha < \sqrt{\pi}e^{-\xi^2/4} < \beta \} + o(\omega).$$

- $k(t) = \frac{1 - \cos t}{t^2}$

Now $a(\xi) = \hat{k}(\xi) = \pi(1 - |\xi|)_+$, $\sigma(\omega C'_{(-1,1)}(\hat{k}\omega))$ fills $[0, \pi]$ densely and, for $0 < \alpha < \beta \leq \pi$, the number of eigenvalues of $C'_{(-1,1)}(\hat{k}\omega)$ in $(\alpha/\omega, \beta/\omega)$ is

$$\frac{\omega}{\pi} \text{mes} \{ \xi : \alpha < \pi(1 - |\xi|)_+ < \beta \} + o(\omega) = \frac{2(\beta - \alpha)}{\pi^2} \omega + o(\omega).$$
ON THE IMPORTANCE OF BEING $L^1(\mathbb{R})$

The Fox–Li kernel $k(t) = e^{it^2}$ is not $L^1(\mathbb{R})$, and this is the source of our problems! Once $k \in L^1(\mathbb{R})$, life becomes simple:

**Theorem** If $k \in L^1(\mathbb{R})$, $k(t) = k(-t)$ and $\mathcal{R}(\hat{k})$ has no interior points, then $\omega \sigma(C_{(-1,1)}(\hat{k}_\omega))$ converges to $\mathcal{R}(\hat{k})$ in the Hausdorff metric and, for every $\varphi \in C(\mathbb{C})$ such that $\lim_{z \to 0} \varphi(z)/z$ is finite, $\varphi \circ a \in L^1(\mathbb{R})$ and

$$
\lim_{\omega \to \infty} \frac{1}{2\omega} \sum_{\lambda \in \sigma(C_{(-1,1)}(\hat{k}_\omega))} \varphi(\omega \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\hat{k}(\xi)) \, d\xi.
$$

**Theorem** Let $k \in L^1(\mathbb{R})$. Then $\omega \sigma(C_{(-1,1)}(\hat{k}_\omega)) \subset \mathcal{R}(|\hat{k}|)$ and converges to it in the Hausdorff metric as $\omega \to \infty$. If $\varphi \in C([0, \infty))$ and $\lim_{x \to 0} \varphi(x)/x$ is finite then $\varphi \circ \hat{k} \in L^1(\mathbb{R})$ and

$$
\lim_{\omega \to \infty} \frac{1}{2\omega} \sum_{\lambda \in \sigma(C_{(-1,1)}(\hat{k}_\omega))} \varphi(\omega \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(|\hat{k}|) \, d\xi.
$$
EXAMPLE: Attenuated Fox–Li.

Let $k(t) = e^{(i - \varepsilon)t^2}$ for $\varepsilon > 0$: 

$$(\mathcal{F}_{\omega, \varepsilon} f)(x) := \int_{-1}^{1} e^{(i - \varepsilon)\omega(x-y)^2} f(y) \, dy, \quad x \in (-1, 1).$$

Since 

$$\hat{k}(\xi) = \sqrt{\frac{\pi(\varepsilon + i)}{1 + \varepsilon^2}} \exp\left(-\frac{\varepsilon \xi^2}{4(1 + \varepsilon^2)}\right) \exp\left(-i\frac{\xi^2}{4(1 + \varepsilon^2)}\right),$$

$\mathcal{R}(\hat{k})$ is a spiral, rotating clockwise from $\frac{\pi(\varepsilon + i)}{1 + \varepsilon^2}$ to the origin. Thus, for $\omega \to \infty$, $\sqrt{\omega} \sigma(\mathcal{F}_{\omega, \varepsilon})$ converges to the spiral 

$$\left\{ \frac{\pi(\varepsilon + i)}{1 + \varepsilon^2} e^{-(i + \varepsilon)t} : t \geq 0 \right\}.$$
Spectra of $\mathcal{F}_{\omega, \varepsilon}$ for $\varepsilon = \frac{1}{4}$ and different values of $\omega$, as well as the spiral $\mathbf{k}$. 
ATTEMPT 1: Wiener–Hopf operators

Although \( F_\omega \) is unitarily equivalent to

\[
\frac{1}{\sqrt{\omega}} W_{2\sqrt{\omega}}(a) \quad \text{with} \quad a(\xi) = \sqrt{\pi} e^{i\pi/4} e^{-i\xi^2/4},
\]

our theory is inapplicable because \( a \not\in C(\mathbb{R}) \cap L^1(\mathbb{R}) \). Consider instead

\[
\ell[\omega](t) := \chi(-2\sqrt{\omega},2\sqrt{\omega})(t)e^{it^2} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).
\]

Since \( \ell[\omega]_\omega(t) := \ell[\omega](\sqrt{\omega}t) = \chi(-2,2)(t)e^{i\omega t^2} \), we deduce that

\[
F_\omega = C_{(-1,1)}\left(\ell[\omega]_\omega\right).
\]

From this it is possible to deduce that

\[
\sqrt{\omega} \sigma(F_\omega) = \sigma(W_{2\sqrt{\omega}}(\ell[\omega])).
\]

The problem is that the right-hand side depends upon \( \omega \) in two different ways.
It is possible to show that \( \sigma(W_\tau(\ell[\omega])) \) is asymptotically distributed along \( R(\ell[\omega]) \) for \( \omega \to \infty \). The snag is that our \( \tau = 2\sqrt{\omega} \) depends upon \( \omega \).

Let us assume (wrongly!) that for \( \omega \gg 1 \) we can replace convolution over \((0, 2\sqrt{\omega})\) by one over \((0, \infty)\). This leads to the ‘conclusion’ that

\[
\sigma(F_\omega) \approx (1/\sqrt{\omega})R(\ell[\omega]).
\]

The spirals \((1/\sqrt{\omega})R(\ell[\omega])\).
ATTEMPT 2: Toeplitz operators

Fix $\omega > 0$ and discretize $F_\omega$ at $2N + 1$ equidistant points in $[-1,1]$. This approximates the spectral problem by the algebraic eigenvalue problem

$$B[N] f[N] = \lambda[N] f[N],$$

where

$$B[N] := (v_{j-k})_{j,k=-N}^{N} \quad \text{with} \quad v[N] := \frac{e^{i\omega n^2/N^2}}{N} \quad \text{is Toepliz.}$$

Given $v \in L^1(T)$ with the Fourier coefficients

$$v_n := \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\theta})e^{-in\theta} \, d\theta, \quad n \in \mathbb{Z},$$

we denote by $T(v)$ and $T_N(v)$ the infinite Toeplitz matrix $(v_{j-k})_{j,k \in \mathbb{Z}_+}$ and finite matrix $B[N] = T_N(v[N])$ resp., where in our case

$$v[N](e^{i\theta}) = \sum_{n=-2N}^{2N} v[n] e^{in\theta} = \frac{1}{N} \sum_{n=-2N}^{2N} e^{i\omega n^2/N^2} e^{in\theta}.$$
Since Fox–Li is compact, \( \sigma(T_N(v[N])) \to \sigma(F_\omega) \), but since both the dimension and the generating function vary with \( N \), we have no theoretical tools to predict the limit of \( \sigma(T_N(v[N])) \). We may again abandon rigour and replace

\[
\sigma(T_N(v[N])) \approx \sigma(T(v[N])) = v[N](\mathbb{T}) \implies \sigma(F_\omega) \approx v[N](\mathbb{T}).
\]

The spirals \( v[N](\mathbb{T}) \) for \( N = 500 \).
The Wiener–Hopf and Toeplitz attempts are equally wrong:

\[
v^{[N]}(e^{i\theta}) = \frac{1}{N} \sum_{n=-2N}^{2N} e^{i\omega n^2/N^2} e^{i\sqrt{\omega} \xi n/N} = \int_{-2}^{2} e^{i\omega x^2} e^{i\sqrt{\omega} \xi x} \, dx + O(1/N) \\
= \frac{1}{\sqrt{\omega}} \int_{-2\sqrt{\omega}}^{2\sqrt{\omega}} e^{it^2} e^{i\xi t} \, dt + O(1/N) = \frac{1}{\sqrt{\omega}} \hat{\ell}^{[\omega]}(\xi) + O(1/N).
\]

Moreover, using asymptotic analysis,

\[
\hat{\ell}^{[\omega]}(\xi) = \frac{\sqrt{\pi} e^{-i\xi^2/4}}{2\sqrt{-i\omega}} \left[ \text{erf} \left( 2\sqrt{-i\omega} + \frac{1}{2} \sqrt{-i\xi} \right) + \text{erf} \left( 2\sqrt{-i\omega} - \frac{1}{2} \sqrt{-i\xi} \right) \right]
\]

therefore

\[
\epsilon^{\text{odd}}(\omega, \xi) = \frac{\sqrt{\pi} e^{-i\xi^2/4}}{2\sqrt{-i\omega}} \left[ \text{erf} \left( 2\sqrt{-i\omega} + \frac{1}{2} \sqrt{-i\xi} \right) + \text{erf} \left( 2\sqrt{-i\omega} - \frac{1}{2} \sqrt{-i\xi} \right) \right] + O(\omega^{-2});
\]

\[
\epsilon^{\text{even}}(\omega, \xi) = \frac{\sqrt{\pi} e^{-i\xi^2/4}}{2\sqrt{-i\omega}} \left[ \text{erf} \left( 2\sqrt{-i\omega} + \frac{1}{2} \sqrt{-i\xi} \right) + \text{erf} \left( 2\sqrt{-i\omega} - \frac{1}{2} \sqrt{-i\xi} \right) \right] + O(\xi^{-2}).
\]

This explains the two regimes visible in the figures: extended rotation with roughly equal amplitude, followed by attenuation.
ATTEMPT 3: Theta-three

Compare

\[
v^{[N]}(e^{2i\alpha}) = \frac{1}{N} \left[ 1 + 2 \sum_{k=1}^{2N} q_n^k \cos(2\alpha k) \right], \quad q_N = e^{i\omega/N^2}, \quad |q_N| = 1,
\]

\[
\theta_3(\alpha|q) := 1 + 2 \sum_{k=1}^{\infty} q^k \cos(2\alpha k), \quad q \in \mathbb{C}, \quad |q| < 1.
\]

What makes \(v^{[N]}\) stay nice, in spite of \(|q_N| = 1\), is the presence of the \(1/N\) factor.

\(\theta_3\) blows up for \(|q| = 1\). Instead, let’s take

\[
q_{N,\omega} = \left(1 - \frac{\sqrt{\omega}}{\sqrt{2N^2}}\right)q_N, \quad 1 > |q_{N,\omega}| = 1 + O(1/N^2)
\]

and plot

\[
\frac{\theta_3(\alpha|q_{N,\omega})}{N} \quad \text{for} \quad N \gg 1, \quad \alpha \in [-\pi/2, \pi/2].
\]
Attenuated theta spirals, superimposed on the spectra.

What is the explanation of this remarkable fit, at least near the ‘head’ of the spiral? We have no idea.