

Analyzing Controllability and Observability by Exploiting the Stratification of Matrix Pairs

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Based on joint work with

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- **The Matrix**
- **The Matrix Reloaded**
- **The Matrix Revolutions**
- **The Matrix Stratifications**

Coming soon to a PC near you!

THEORY – ALGORITHMS – SOFTWARE TOOLS

- **Theme 1:** Matrix Pencil Computations in Computer-Aided Control System Design
 - Ill-posed eigenvalue problems
 - Canonical forms (Jordan, Kronecker, staircase)
 - Generalized Schur forms (GUPTRI, QZ)
 - Subspaces: eigenvalue reordering
 - Matrix equations (Sylvester, Lyapunov, Riccati)
 - Functions of matrices
 - Perturbation theory, condition estimation and error bounds
 - Periodic counterparts
 - Periodic Riccati differential equations
- **Theme 2:** High Performance and Parallel Computing

THEORY – ALGORITHMS – SOFTWARE TOOLS

- **Theme 2:** High Performance and Parallel Computing
 - Blocking for memory hierarchies (DM, SM, hybrid, multicore, GPGPUs)
 - Explicit (multi level) blocking
 - Recursive blocking
 - Blocked hybrid data structures
 - Library software
 - Contributions to LAPACK, ScaLAPACK, SLICOT, ESSL
 - Matrix equations: RECSY and SCASY
 - Novel parallel QR algorithm
 - 30 times faster than current ScaLAPACK implementation!
 - Solved 100000×100000 dense nonsymmetric eigenvalue problems!

- Some motivation and background to stratification of orbits and bundles:
 - Canonical forms and structure information
 - Matrix and pencil spaces
 - Graph representation of a closure hierarchy
 - Nilpotent matrix orbit stratification (7×7)
 - Matrix bundle stratification (4×4)
- Controllability and observability matrix pairs
 - System pencils and equivalence orbits and bundles
 - Canonical forms of pairs (Kronecker and Brunovsky)
 - Closure and cover relations
- Applications in control system design and analysis:
 - Mechanical system - uniform platform with 2 springs
 - Linearized Boeing 747 model

Stratification [*Oxford advanced learner's dictionary*]

The division of something into different layers or groups

- Computation of canonical forms (e.g., Jordan, Kronecker, Brunovsky) are ill-posed problems
 - small perturbation of input data may drastically change the computed structure
- Compute canonical structure information using so called staircase algorithms (orthogonal transformations)
- Need to provide the user with more information:
 - What other structures are nearby?
 - Upper and lower bounds to other structures
- Applications in, e.g., control system design
 - Controllability
 - Observability

- *Objective:* Make use of the **geometry of matrix and matrix pencil spaces** to **solve nearness problems** related to Jordan and Kronecker canonical forms
- *Tools:* The **theory of stratification** of orbits and bundles (and versal deformations)

Our program:

To understand **qualitative and quantitative properties** of nearby Jordan and Kronecker structures

- *Deliveries:* **Interactive tools and algorithms** that make these complex theories easily available to end users

Matrix and matrix pencil spaces

- An $n \times n$ matrix can be viewed as a point in n^2 -dim space
- Numerical computations – move from point to point or manifold to manifold

Orbit of a matrix

$$\mathcal{O}(A) = \{PAP^{-1} : \det P \neq 0\}$$

Manifold of all matrices with Jordan Normal Form (JNF) of A

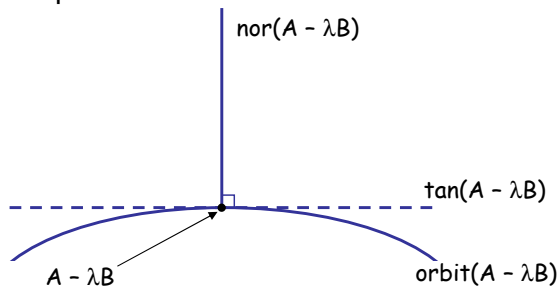
Orbit of a pencil

$$\mathcal{O}(A - \lambda B) = \{P(A - \lambda B)Q : \det P \det Q \neq 0\}$$

Manifold of all $m \times n$ pencils in $2mn$ -dim space with the Kronecker Canonical Form (KCF) of $A - \lambda B$

- **Bundle:** $\mathcal{B}(\cdot)$ –union of all orbits with the same canonical form but with eigenvalues unspecified

$m \times n$ pencil case



- $\dim(\mathcal{O}(A - \lambda B)) = \dim(\text{tan}(\mathcal{O}(A - \lambda B)))$
- $\text{codim}(\mathcal{O}(A - \lambda B)) = \dim(\text{nor}(\mathcal{O}(A - \lambda B)))$
- $\dim(\mathcal{O}(A - \lambda B)) + \text{codim}(\mathcal{O}(A - \lambda B)) = 2mn$
- $\text{codim}(\mathcal{B}(\cdot)) = \text{codim}(\mathcal{O}(\cdot)) - k$,
 $k = \text{number of unspecified eigenvalues}$

- Given a matrix and its orbit: What other structures are found within its closure?

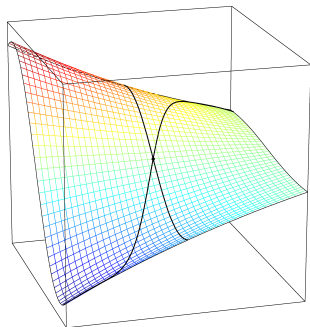
Stratification:

The closure hierarchy of all possible Jordan structures

We make use of:

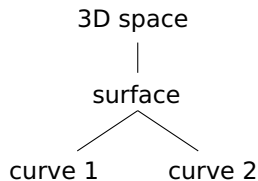
- Graphs to illustrate stratifications
- Dominance orderings for integer partitions in proofs and derivations

Closure hierarchy – graph representation

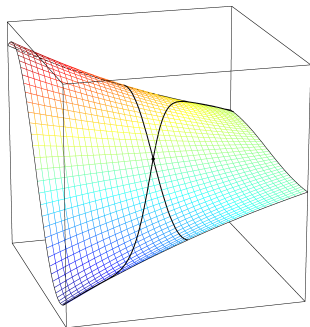


The 3D space covers the surface

The two curves are in the **closure** of the surface, which is in the closure of the 3D space



Closure hierarchy – graph representation



Most generic

3D space

surface

curve 1

curve 2

Least generic
(most
degenerate)

point

Staircase form of nilpotent 7×7 matrix

$$\left[\begin{array}{ccc|cc|cc} \overbrace{0 \ 0 \ 0}^{m_1} & \overbrace{x \ x}^{m_2} & \overbrace{x \ x}^{m_3} & & & & \\ & x \ x & x \ x & & & & \\ & 0 \ 0 & x \ x & & & & \\ & 0 & x \ x & & & & \\ \hline & 0 \ 0 & y \ y & & & & \\ & 0 & y \ y & & & & \\ \hline & & 0 \ 0 & & & & \\ & & & & & & 0 \end{array} \right]$$

Weyr: $(3, 2, 2)$

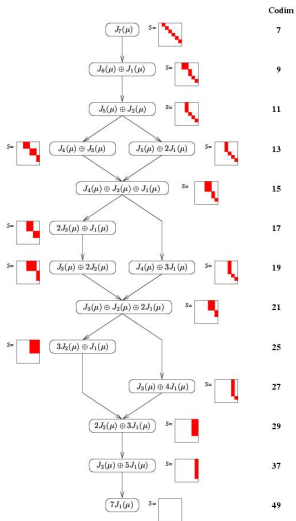
$$\begin{aligned} m_1 &= 3 = \dim \mathcal{N}(A - \mu I), \\ m_1 + m_2 &= 5 = \dim \mathcal{N}((A - \mu I)^2), \\ m_1 + m_2 + m_3 &= 7 = \dim \mathcal{N}((A - \mu I)^3) \end{aligned}$$

$$J_3(0) \oplus J_3(0) \oplus J_1(0)$$

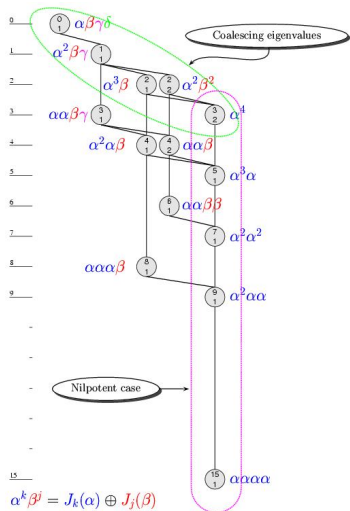
Segre: $(3, 3, 1)$

Nilpotent orbit stratification of 7×7 matrix

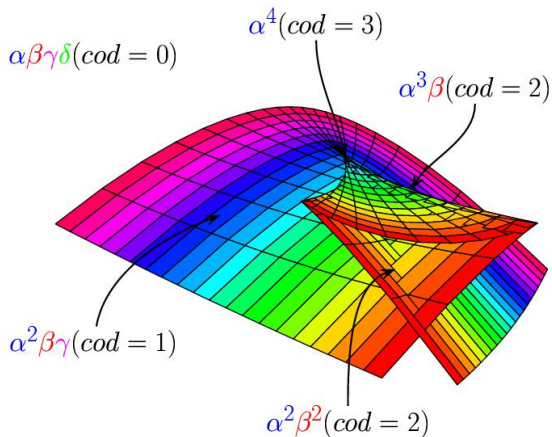
- Dominance ordering of the integer $n = 7$
- Deformations of stairs and
- ... versal deformations
- $A + Z(p)$
 $\text{span}(Z(p)) = \text{nor}(A)$



Stratification of 4×4 matrix bundle



Swallowtail – bundles of coalescing eigenvalues



A (generalized) state-space system with the *state-space model*

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

can be represented in the form of a *system pencil*

$$\mathbf{S}(\lambda) = G - \lambda H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},$$

with the corresponding general *matrix pencil* $G - \lambda H$

In short form, $\mathbf{S}(\lambda)$ is represented by a *matrix quadruple* (A, B, C, D) ($E = I$)

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

- **Controllable system:** There exists an input signal (vector) $u(t)$, $t_0 \leq t \leq t_f$ that takes every state variable from an initial state $x(t_0)$ to a desired final state x_f in finite time.
- **Observable system:** If it possible to find the initial state $x(t_0)$ from the input signal $u(t)$ and the output signal $y(t)$ measured over a finite interval $t_0 \leq t \leq t_f$.

Consider the *controllability pair* (A, B) and the *observability pair* (A, C) , associated with the particular systems:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$\dot{x}(t) = Ax(t)$$

$$y(t) = Cx(t)$$

System pencil representations:

$$\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I_n \ 0] \quad \text{and} \quad \mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

An orbit of a matrix pair

A manifold of equivalent matrix pairs:

$$\mathcal{O}(A, B) = \left\{ P(A, B) \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det P \cdot \det Q \neq 0 \right\}$$

A **bundle** of a matrix pair

The union of all orbits with the same canonical form but with unspecified eigenvalues

$$\mathcal{B}(A, B) = \bigcup_{\mu_i} \mathcal{O}(A, B)$$

A canonical form is the simplest or most symmetrical form a matrix or matrix pencil can be reduced to

- Matrices – [Jordan canonical form](#)
- Matrix pencils – [Kronecker canonical form](#)
- System pencils – [\(generalized\) Brunovsky canonical form](#)

A canonical form reveals the canonical structure information from which the system characteristics are deduced

All matrix pairs in the **same orbit** has the **same canonical form**

Kronecker canonical form

Any matrix pencil $G - \lambda H$ or system pencil $\mathbf{S}(\lambda)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations (U and V non-singular):

$$U^{-1}(\mathbf{S}(\lambda))V =$$

$$\text{diag}(L_{\epsilon_1}, \dots, L_{\epsilon_p}, J(\mu_1), \dots, J(\mu_t), N_{s_1}, \dots, N_{s_k}, L_{\eta_1}^T, \dots, L_{\eta_q}^T)$$

Singular part:

- $L_{\epsilon_1}, \dots, L_{\epsilon_p}$ – Right singular blocks
- $L_{\eta_1}^T, \dots, L_{\eta_q}^T$ – Left singular blocks

$$L_k = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}$$

Any matrix pencil $G - \lambda H$ or system pencil $\mathbf{S}(\lambda)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations (U and V non-singular):

$$U^{-1}(\mathbf{S}(\lambda))V =$$

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Singular part:

- $L_{\epsilon_1}, \dots, L_{\epsilon_p}$ – Right singular blocks
- $L_{\eta_1}^T, \dots, L_{\eta_q}^T$ – Left singular blocks

$$J_k(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 1 \\ & & & & & & \mu_i - \lambda \end{bmatrix}$$

Regular part:

- $J(\mu_1), \dots, J(\mu_t)$ – Each $J(\mu_i)$ is block-diagonal with *Jordan* blocks corresponding to the finite eigenvalue μ_i
- N_{s_1}, \dots, N_{s_k} – Jordan blocks corresponding to the infinite eigenvalue

- $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix}$ has full row rank \Rightarrow KCF of $\mathbf{S}_C(\lambda)$ can only have finite eigenvalues (uncontrollable modes) and L_k blocks:

$$U^{-1}\mathbf{S}_C(\lambda)V = \text{diag}(L_{\epsilon_1}, \dots, L_{\epsilon_p}, J(\mu_1), \dots, J(\mu_t))$$

- $\mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ has full column rank \Rightarrow KCF of $\mathbf{S}_O(\lambda)$ can only have finite eigenvalues (unobservable modes) and L_k^T blocks:

$$U^{-1}\mathbf{S}_O(\lambda)V = \text{diag}(J(\mu_1), \dots, J(\mu_t), L_{\eta_1}^T, \dots, L_{\eta_p}^T)$$

Brunovsky canonical form

Given a matrix pair (A, B) or (A, C) , there exists a *feedback equivalent* matrix pair in *Brunovsky canonical form* (BCF), such that

$$P \begin{bmatrix} A - \lambda I_n & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} = \left[\begin{array}{cc|c} A_\epsilon & 0 & B_\epsilon \\ 0 & A_\mu & 0 \end{array} \right]$$

or

$$\begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} P^{-1} = \left[\begin{array}{cc|c} A_\eta & 0 & \\ 0 & A_\mu & \\ \hline \bar{C}_\eta & & 0 \end{array} \right]$$

respectively, where

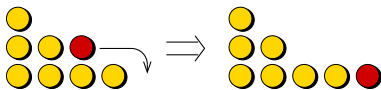
- (A_ϵ, B_ϵ) – *controllable* and corresponds to the L blocks
- (A_η, C_η) – *observable* and corresponds to the L^T blocks
- A_μ – block diagonal with Jordan blocks and corresponds to the *uncontrollable* and *unobservable eigenvalues*, respectively

- $\mathcal{R} = (r_0, r_1, \dots)$ where $r_i = \#L_k$ blocks with $k \geq i$
- $\mathcal{L} = (l_0, l_1, \dots)$ where $l_i = \#L_k^T$ blocks with $k \geq i$
- $\mathcal{J}_{\mu_i} = (j_1, j_2, \dots)$ where $j_i = \#J_k(\mu_i)$ blocks with $k \geq i$.
 \mathcal{J}_{μ_i} is known as the *Weyr characteristics* of the finite eigenvalue μ_i
- $\mathcal{N} = (n_1, n_2, \dots)$ where $n_i = \#N_k$ with $k \geq i$. \mathcal{N} is known as the *Weyr characteristics* of the infinite eigenvalue

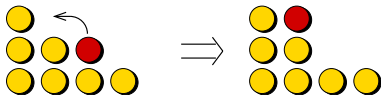
Integer partitions

A *partition* ν of an integer K is defined as $\nu = (\nu_1, \nu_2, \dots)$ where $\nu_1 \geq \nu_2 \geq \dots \geq 0$ and $K = \nu_1 + \nu_2 + \dots$

Minimum rightward coin move: rightward *one* column or downward *one* row (keep partition monotonic)



Minimum leftward coin move: leftward *one* column or upward *one* row (keep partition monotonic)



[Edelman, Elmroth & Kågström; 1999]

Theorem

Given the structure integer partitions \mathcal{R} and \mathcal{J}_{μ_i} of (A, B) , one of the following if-and-only-if rules finds (\tilde{A}, \tilde{B}) such that:

$\mathcal{O}(A, B)$ **covers** $\mathcal{O}(\tilde{A}, \tilde{B})$

- 1 Minimum rightward coin move in \mathcal{R}
- 2 If the *rightmost column in \mathcal{R} is one single coin*, move that coin to a new rightmost column of some \mathcal{J}_{μ_i} (which may be empty initially)
- 3 Minimum leftward coin move in any \mathcal{J}_{μ_i}

Rules 1 and 2: Coin moves that affect r_0 are not allowed

$\mathcal{O}(A, B)$ **is covered by** $\mathcal{O}(\tilde{A}, \tilde{B})$

- 1 Minimum leftward coin move in \mathcal{R} , without affecting r_0
- 2 If the *rightmost column in some \mathcal{J}_{μ_i} consists of one coin only*, move that coin to a new rightmost column in \mathcal{R}
- 3 Minimum rightward coin move in any \mathcal{J}_{μ_i}

Theorem

Given the structure integer partitions \mathcal{R} and \mathcal{J}_{μ_i} of (A, B) , one of the following if-and-only-if rules finds (\tilde{A}, \tilde{B}) such that:

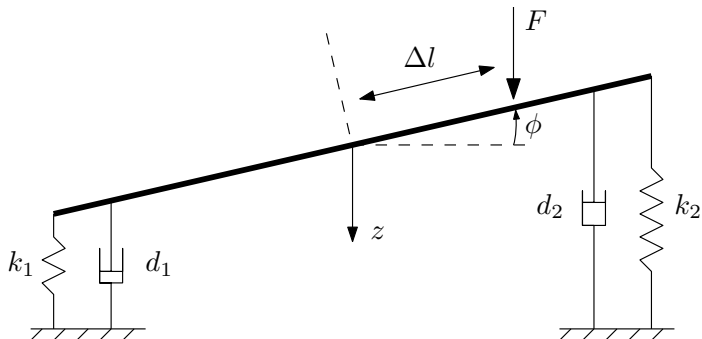
$\mathcal{B}(A, B)$ **covers** $\mathcal{B}(\tilde{A}, \tilde{B})$

- 1 Minimum rightward coin move in \mathcal{R}
- 2 If the rightmost column in \mathcal{R} is one single coin, move that coin to the first column of \mathcal{J}_{μ_i} for a new eigenvalue μ_i
- 3 Minimum leftward coin move in any \mathcal{J}_{μ_i}
- 4 Let any pair of eigenvalues coalesce, i.e., **take the union** of their sets of coins

$\mathcal{B}(A, B)$ **is covered by** $\mathcal{B}(\tilde{A}, \tilde{B})$

- 1 Minimum leftward coin move in \mathcal{R} , without affecting r_0
- 2 If some \mathcal{J}_{μ_i} consists of one coin only, move that coin to a new rightmost column in \mathcal{R}
- 3 Minimum rightward coin move in any \mathcal{J}_{μ_i}
- 4 For any \mathcal{J}_{μ_i} , **divide the set** of coins into two new sets so that their union is \mathcal{J}_{μ_i}

Example 1 – Mechanical system



A **uniform platform** with mass m and length $2l$, supported in both ends by springs

The **control parameter** of the system is the force F applied at distance Δl from the center of the platform

By linearizing the equations of motion near the equilibrium the system can be expressed by the linear state-space model $\dot{x} = Ax(\tau) + Bu(\tau)$ [A. Mailybaev '03]:

$$\begin{bmatrix} \omega \dot{z}/l \\ \omega \dot{\phi} \\ \omega^2 \ddot{z}/l \\ \omega^2 \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1 & -c_2 & -f_1 & -f_2 \\ -3c_2 & -3c_1 & -3f_2 & -3f_1 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega \dot{z}/l \\ \omega \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3\Delta \end{bmatrix} \frac{\omega^2}{ml} F$$

where

Fixed elements!

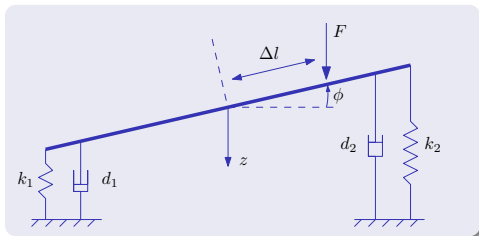
$$c_1 = \frac{(k_1 + k_2)\omega^2}{m}, \quad c_2 = \frac{(k_1 - k_2)\omega^2}{m},$$
$$f_1 = \frac{(d_1 + d_2)\omega}{m}, \quad f_2 = \frac{(d_1 - d_2)\omega}{m},$$

and $\tau = t/\omega$ where ω is a time scale coefficient

Mechanical system – Canonical forms

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ -0.0004 & 0 & -0.027 & 0 & | & 1 \\ 0 & -0.0012 & 0 & -0.08 & | & 0 \end{bmatrix}$$
$$-\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$



With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$\mathbf{U} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.0004 & 0 & -0.027 & 0 & 1 \\ 0 & -0.0012 & 0 & -0.08 & 0 \end{array} \right] \\
 -\lambda \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \mathbf{V}^{-1}$$

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$\begin{bmatrix}
 \boxed{0} & \boxed{1} & \boxed{0} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{1} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-0.02} & \boxed{0} & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-0.06} & 0
 \end{bmatrix}$$

$$-\lambda \begin{bmatrix}
 \boxed{1} & \boxed{0} & \boxed{0} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{1} & \boxed{0} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & 0 & \boxed{1} & 0
 \end{bmatrix}
 = L_2 \oplus J_1(-0.02) \oplus J_1(-0.06)$$

Kronecker canonical form

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$\mathbf{P}_{\text{row}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.06 & 0 \end{bmatrix} \\
 -\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{P}_{\text{col}}$$

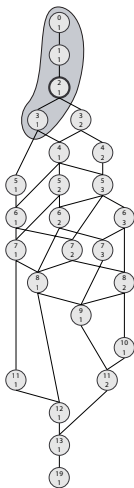
With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & -0.02 & 0 & | & 0 \\ 0 & 0 & 0 & -0.06 & | & 0 \end{bmatrix}$$

$$-\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Uncontrollable eigenvalues (modes)

Brunovsky canonical form

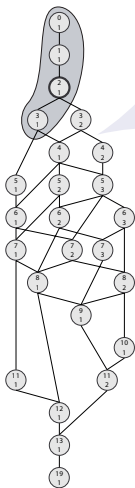


The software tool **StratiGraph** is used for computing and visualizing the stratification

[Elmroth, P. Johansson & Kågström; 2001]

[P. Johansson; PhD Thesis 2006]

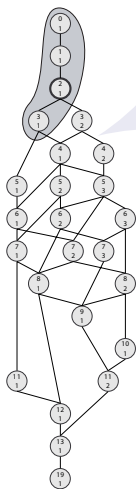
Mechanical system – Illustrating the bundle stratification



Each **edge** represents a cover relation

It is always possible to go from any canonical structure (node) to another higher up in the graph by a **small perturbation**

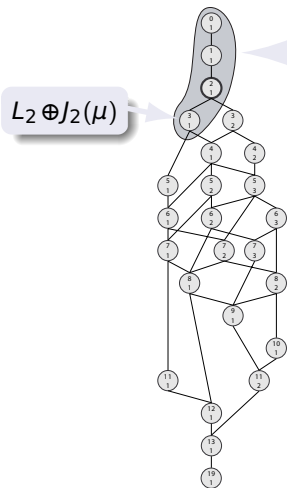
Mechanical system – Illustrating the bundle stratification



Each **edge** represents a cover relation

A **cover relation** is determined by the **combinatorial rules** acting on the integer sequences representing the canonical structure information

Mechanical system – Illustrating the bundle stratification



$L_2 \oplus J_2(\mu)$

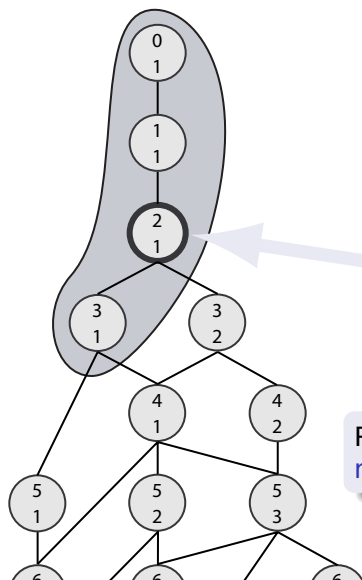
Only possible bundles for the example

Least generic bundle

Let all **free elements** in the system matrices be zero \Rightarrow The **least generic possible bundle** has the KCF $L_2 \oplus J_2(\mu)$

$$\begin{bmatrix} \omega \dot{z}/l \\ \omega \dot{\phi} \\ \omega^2 \ddot{z}/l \\ \omega^2 \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega \dot{z}/l \\ \omega \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{\omega^2}{ml} F$$

Mechanical system – Illustrating the bundle stratification



$L_2 \oplus$

\mathcal{R} : ●●●

$J_1(-0.02) \oplus$

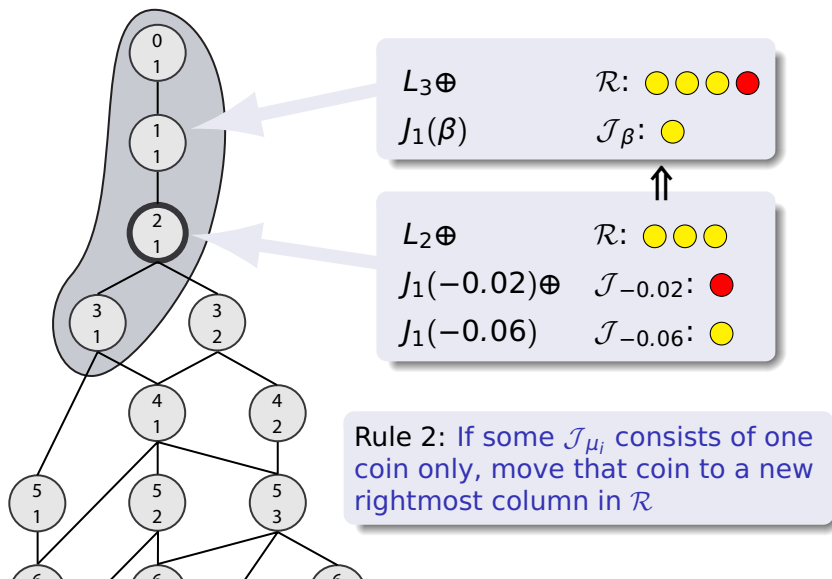
$\mathcal{J}_{-0.02}$: ●

$J_1(-0.06)$

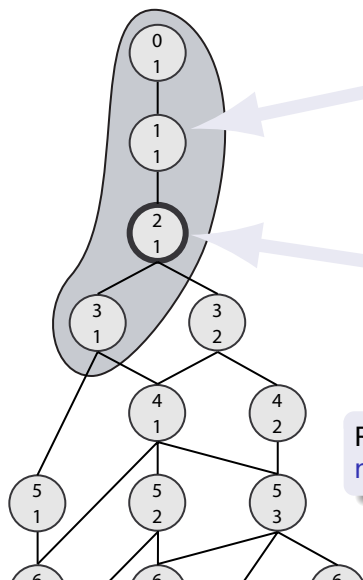
$\mathcal{J}_{-0.06}$: ●

Rule 1: Minimum leftward coin move in \mathcal{R} , without affecting r_0

Mechanical system – Illustrating the bundle stratification



Mechanical system – Illustrating the bundle stratification



$L_3 \oplus$

$J_1(\beta)$

$L_2 \oplus$

$J_1(-0.02) \oplus$

$J_1(-0.06)$

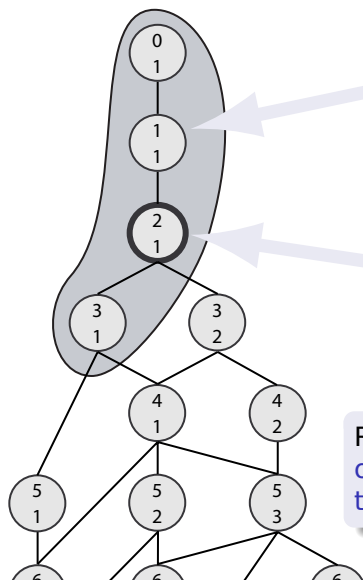
\mathcal{R} : ● ● ●

$\mathcal{J}_{-0.02}$: ●

$\mathcal{J}_{-0.06}$: ●

Rule 3: Minimum rightward coin
move in any \mathcal{J}_{μ_i}

Mechanical system – Illustrating the bundle stratification



$L_3 \oplus$

$J_1(\beta)$

$L_2 \oplus$

\mathcal{R} : ● ● ●

$J_1(-0.02) \oplus$

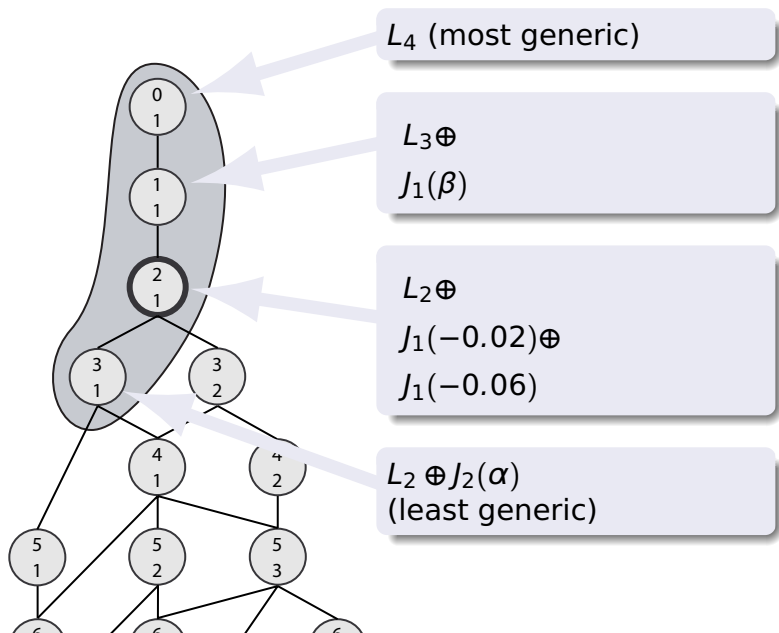
$\mathcal{J}_{-0.02}$: ●

$J_1(-0.06)$

$\mathcal{J}_{-0.06}$: ●

Rule 4: For any \mathcal{J}_{μ_i} , divide the set of coins into two new sets so that their union is \mathcal{J}_{μ_i}

Mechanical system – Illustrating the bundle stratification



Example 2 – Boeing 747

A Boeing 747 under straight-and-level flight at **altitude** 600 m with **speed** 92.6 m/s, **flap setting** at 20° , and landing gears up. The aircraft has **mass** = 317,000 kg and the **center of gravity coordinates** are $X_{cg} = 25\%$, $Y_{cg} = 0$, and $Z_{cg} = 0$



Boeing 747 – State-space model

A linearized nominal longitudinal model with 5 states and 5 inputs [A. Varga '07]:

$$\dot{x} = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} u(t)$$

$$x = \begin{bmatrix} \delta q \\ \delta V_{TAS} \\ \delta \alpha \\ \delta \theta \\ \delta h_e \end{bmatrix} \begin{pmatrix} \text{pitch rate (rad/s)} \\ \text{true airspeed (m/s)} \\ \text{angle of attack (rad)} \\ \text{pitch angle (rad)} \\ \text{altitude (m)} \end{pmatrix}, \quad u = \begin{bmatrix} \delta e_i \\ \delta e_o \\ \delta i_h \\ \delta EPR_{1,4} \\ \delta EPR_{2,3} \end{bmatrix} \begin{pmatrix} \text{total inner elevator (rad)} \\ \text{total outer elevator (rad)} \\ \text{stabilizer trim angle (rad)} \\ \text{total thrust engine \#1 and \#4 (rad)} \\ \text{total thrust engine \#2 and \#3 (rad)} \end{pmatrix}$$

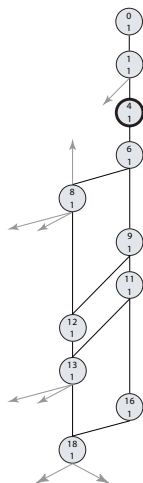
Goal

Find all *possible* **closest uncontrollable systems** which can be reached by a perturbation of the system matrices, and **distance bounds to uncontrollability**

Means:

- 1 Identify all the controllable and the nearest uncontrollable systems in the **orbit stratification**
- 2 Determine the **most and least generic orbits** by considering the **structural restrictions** of the system matrices

Boeing 747 – Illustrating the orbit stratification

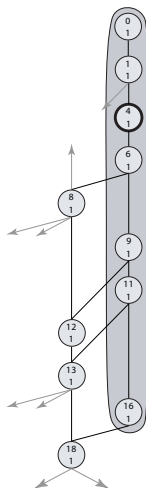


Complete orbit stratification:

- 74 nodes and 133 edges
 - Ranges from codimension 0 to 50
- ⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

Boeing 747 – Illustrating the orbit stratification



Complete orbit stratification:

- 74 nodes and 133 edges
 - Ranges from codimension 0 to 50
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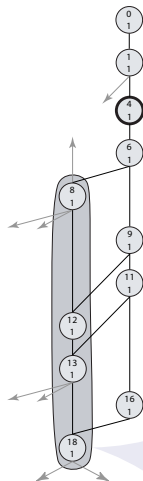
Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

Nodes corresponding to all controllable systems

Boeing 747 – Illustrating the orbit stratification

Complete orbit stratification:

- 74 nodes and 133 edges
 - Ranges from codimension 0 to 50
- ⇒ Identify only the nodes of interest!



Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

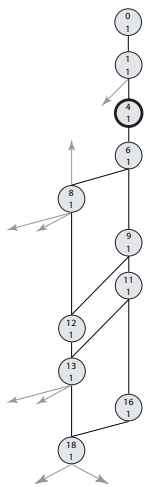
Nodes corresponding to the nearest uncontrollable systems (J_1 -block)

Boeing 747 – Illustrating the orbit stratification

NOT POSSIBLE!

$5L_1$

$L_2 \oplus 3L_1 \oplus L_0$

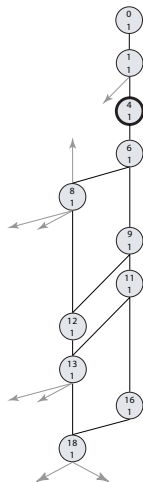


Most generic orbit

Let all free elements in the system matrices be **nonzero** \Rightarrow
 $\#L_0$ blocks = $5 - \text{rank}(B)$, i.e., the most generic orbit must have at least two L_0 blocks

$$A = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Least generic orbit

Let all **free elements** in the system matrices be **zero** \Rightarrow

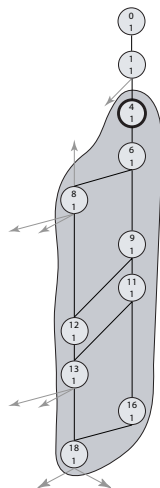
The least generic orbit has the KCF $5L_0 \oplus J_2(\alpha) \oplus 3J_1(\beta)$ with codimension 42

No state can be controlled by any of the inputs!

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

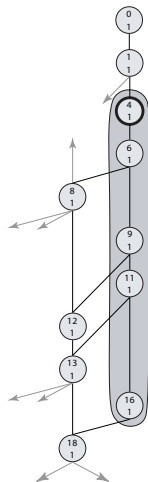
$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Boeing 747 – Illustrating the orbit stratification



Possible orbits of interest

Boeing 747 – Controllable orbits



4: $2L_2 \oplus L_1 \oplus 2L_0$

\tilde{u}_1 controls \tilde{x}_1, \tilde{x}_2 ; \tilde{u}_2 controls \tilde{x}_3, \tilde{x}_4 ;
 \tilde{u}_3 controls \tilde{x}_5

6: $L_3 \oplus 2L_1 \oplus 2L_0$

\tilde{u}_1 controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; \tilde{u}_2 controls \tilde{x}_4 ;
 \tilde{u}_3 controls \tilde{x}_5

9: $L_3 \oplus L_2 \oplus 3L_0$

\tilde{u}_1 controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; \tilde{u}_2 controls \tilde{x}_4, \tilde{x}_5 ;

11: $L_4 \oplus L_1 \oplus 3L_0$

\tilde{u}_1 controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$; \tilde{u}_2 controls \tilde{x}_5 ;

16: $L_5 \oplus 4L_0$

\tilde{u}_1 controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5$;

Given: $m \times n$ pencil $G - \lambda H$

Find: upper and lower bounds on the distance to the closest pencil (say $K - \lambda L$) with a specified KCF

Upper bound:

- Find perturbations $(\delta G, \delta H)$ such that $(G + \delta G) - \lambda(H + \delta H)$ has the KCF of $K - \lambda L$
- $(\delta G, \delta H)$ computed by a staircase algorithm that imposes the specified canonical structure (iGUPTRI)
- $\|(\delta G, \delta H)\|_F$ gives the upper bound

Lower bound:

- Use characterization of tangent space $\tan(G - \lambda H)$ of the orbit:

$$(XG - GY) - \lambda(XH - HY), \quad \forall X, Y$$

- Now, $\tan(G - \lambda H)$ is the range of T , where

$$T \equiv \begin{bmatrix} G^T \otimes I_m & -I_n \otimes G \\ H^T \otimes I_m & -I_n \otimes H \end{bmatrix}$$

- Given $c = \text{cod}(G - \lambda H)$, a lower bound to a pencil $(G + \delta G) - \lambda(H + \delta H)$ with codimension $c + d$ is

$$\|(\delta G, \delta H)\|_F \geq \frac{1}{\sqrt{m+n}} \left(\sum_{i=2mn-c-d+1}^{2mn} \sigma_i(T) \right)$$

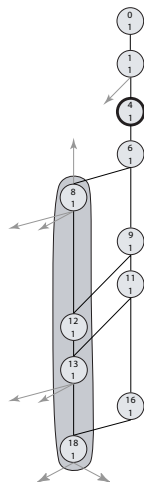
where $\sigma_i(T) \geq \sigma_{i+1}(T)$

Similar characterizations give lower bounds for matrix pairs with tangent space represented as

$$T_{(A,B)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & I_n \otimes B & 0 \\ B^T \otimes I_n & 0 & I_m \otimes B \end{bmatrix} \quad \text{and}$$

$$T_{(A,C)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & C^T \otimes I_n & 0 \\ -I_n \otimes C & 0 & C^T \otimes I_p \end{bmatrix}$$

Matrix case: $T_A = I_n \oplus A - A^T \oplus I_n$

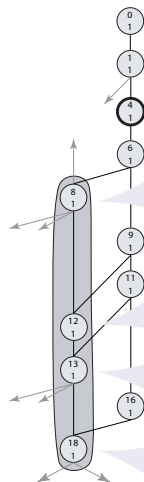


Distance to uncontrollability

$$\tau(A, B) = \min \left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\| : \right. \\ \left. (A + \Delta A, B + \Delta B) \text{ is uncontrollable} \right\}$$

where $\| \cdot \|$ denotes the 2-norm or Frobenius norm

Boeing 747 – Illustrating the orbit stratification



Computed distance to uncontrollability
[Gu et al., 2006]: $3.03e-2$

Lower bound: $4.33e-4$
Upper bound: 1.0

Lower bound: $1.09e-3$
Upper bound: $2.48e-1$

Lower bound: $1.33e-3$
Upper bound: $1.79e-1$

Lower bound: $7.57e-2$
Upper bound: $5.56e-1$

Papers

- E. Elmroth, S. Johansson, and B. Kågström
Stratification of Controllability and Observability Pairs – Theory and Use in Applications. *SIAM J. Matrix Analysis and Applications*, Vol. 31, No. 2, 2009
- S. Johansson
Reviewing the Closure Hierarchy of Orbits and Bundles of System Pencils and Their Canonical Forms. Report UMINF-09.02, Umeå University, 2009
- E. Elmroth, P. Johansson, and B. Kågström
Bounds for the Distance Between Nearby Jordan and Kronecker Structures in a Closure Hierarchy. *Journal of Mathematical Sciences*, Vol. 114, No. 6, 2003
- A. Edelman, E. Elmroth, and B. Kågström
A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils. Part II: A Stratification-Enhanced Staircase Algorithm. *SIAM J. Matrix Analysis and Applications*, Vol. 20, No. 3, 1999
- A. Edelman, E. Elmroth, and B. Kågström
A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils. Part I: Versal Deformations. *SIAM J. Matrix Analysis and Applications*, Vol. 18, No. 3, 1997
(awarded **SIAM/SIAG Linear Algebra Prize 2000**)

PhD Theses

- S. Johansson
Tools for Control System Design – Stratification of Matrix Pairs and Periodic Riccati Differential Equation Solvers. Department of Computing Science, Umeå University, 2009
- P. Johansson
Software Tools for Matrix Canonical Computations and Web-Based Software Library Environments. Department of Computing Science, Umeå University, 2006

- While **stratigraphy** is the key to understanding the geological evolution of the world, **StratiGraph** is the entry to understanding the "geometrical evolution" of orbits and bundles in the "world" of matrices and matrix pencils.
- But remember these worlds grow exponentially with matrix size!
- Thanks!