

# Analyzing Controllability and Observability by Exploiting the Stratification of Matrix Pairs

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Based on joint work with

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- **The Matrix**
- **The Matrix Reloaded**
- **The Matrix Revolutions**
- **The Matrix Stratifications**

**Coming soon to a PC near you!**

## THEORY – ALGORITHMS – SOFTWARE TOOLS

- **Theme 1:** Matrix Pencil Computations in Computer-Aided Control System Design
  - Ill-posed eigenvalue problems
  - Canonical forms (Jordan, Kronecker, staircase)
  - Generalized Schur forms (GUPTRI, QZ)
  - Subspaces: eigenvalue reordering
  - Matrix equations (Sylvester, Lyapunov, Riccati)
  - Functions of matrices
  - Perturbation theory, condition estimation and error bounds
  - Periodic counterparts
    - Periodic Riccati differential equations
- **Theme 2:** High Performance and Parallel Computing

## THEORY – ALGORITHMS – SOFTWARE TOOLS

- **Theme 2:** High Performance and Parallel Computing
  - Blocking for memory hierarchies (DM, SM, hybrid, multicore, GPGPUs)
    - Explicit (multi level) blocking
    - Recursive blocking
    - Blocked hybrid data structures
  - Library software
    - Contributions to LAPACK, ScaLAPACK, SLICOT, ESSL
    - Matrix equations: RECSY and SCASY
  - Novel parallel QR algorithm
    - 30 times faster than current ScaLAPACK implementation!
    - Solved  $100000 \times 100000$  dense nonsymmetric eigenvalue problems!

- Some motivation and background to stratification of orbits and bundles:
  - Canonical forms and structure information
  - Matrix and pencil spaces
  - Graph representation of a closure hierarchy
  - Nilpotent matrix orbit stratification ( $7 \times 7$ )
  - Matrix bundle stratification ( $4 \times 4$ )
- Controllability and observability matrix pairs
  - System pencils and equivalence orbits and bundles
  - Canonical forms of pairs (Kronecker and Brunovsky)
  - Closure and cover relations
- Applications in control system design and analysis:
  - Mechanical system - uniform platform with 2 springs
  - Linearized Boeing 747 model

Stratification [*Oxford advanced learner's dictionary*]

The division of something into different layers or groups

- Computation of canonical forms (e.g., Jordan, Kronecker, Brunovsky) are ill-posed problems
  - small perturbation of input data may drastically change the computed structure
- Compute canonical structure information using so called staircase algorithms (orthogonal transformations)
- Need to provide the user with more information:
  - What other structures are nearby?
  - Upper and lower bounds to other structures
- Applications in, e.g., control system design
  - Controllability
  - Observability

- *Objective:* Make use of the **geometry of matrix and matrix pencil spaces** to **solve nearness problems** related to Jordan and Kronecker canonical forms
- *Tools:* The **theory of stratification** of orbits and bundles (and versal deformations)

## *Our program:*

To understand **qualitative and quantitative properties** of nearby Jordan and Kronecker structures

- *Deliveries:* **Interactive tools and algorithms** that make these complex theories easily available to end users

# Matrix and matrix pencil spaces

- An  $n \times n$  matrix can be viewed as a point in  $n^2$ -dim space
- Numerical computations – move from point to point or manifold to manifold

## Orbit of a matrix

$$\mathcal{O}(A) = \{PAP^{-1} : \det P \neq 0\}$$

Manifold of all matrices with Jordan Normal Form (JNF) of  $A$

## Orbit of a pencil

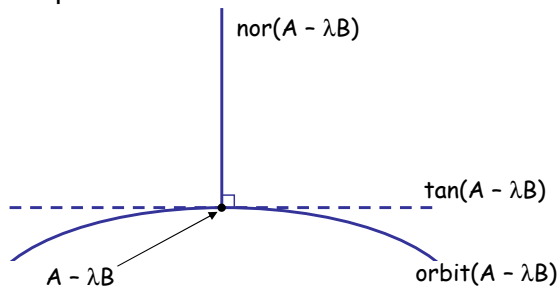
$$\mathcal{O}(A - \lambda B) = \{P(A - \lambda B)Q : \det P \det Q \neq 0\}$$

Manifold of all  $m \times n$  pencils in  $2mn$ -dim space with the Kronecker Canonical Form (KCF) of  $A - \lambda B$

- **Bundle:**  $\mathcal{B}(\cdot)$ –union of all orbits with the same canonical form but with eigenvalues unspecified



$m \times n$  pencil case



- $\dim(\mathcal{O}(A - \lambda B)) = \dim(\text{tan}(\mathcal{O}(A - \lambda B)))$
- $\text{codim}(\mathcal{O}(A - \lambda B)) = \dim(\text{nor}(\mathcal{O}(A - \lambda B)))$
- $\dim(\mathcal{O}(A - \lambda B)) + \text{codim}(\mathcal{O}(A - \lambda B)) = 2mn$
- $\text{codim}(\mathcal{B}(\cdot)) = \text{codim}(\mathcal{O}(\cdot)) - k$ ,  
 $k = \text{number of unspecified eigenvalues}$

- Given a matrix and its orbit: What other structures are found within its closure?

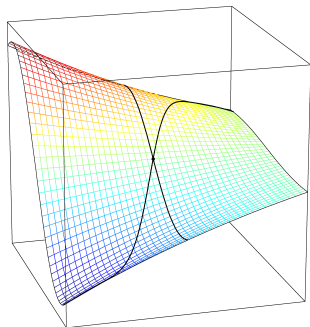
## *Stratification:*

The closure hierarchy of all possible Jordan structures

We make use of:

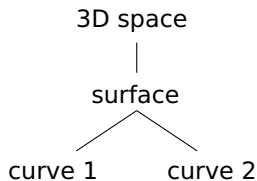
- Graphs to illustrate stratifications
- Dominance orderings for integer partitions in proofs and derivations

# Closure hierarchy – graph representation

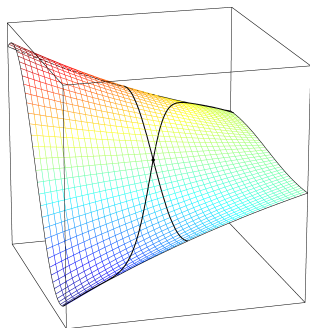


The 3D space covers the surface

The two curves are in the **closure** of the surface, which is in the closure of the 3D space



# Closure hierarchy – graph representation



Most generic

3D space

surface

curve 1

curve 2

Least generic  
(most  
degenerate)

point

# Staircase form of nilpotent $7 \times 7$ matrix

$$\begin{array}{c|c|c}
 \overbrace{\begin{matrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{matrix}}^{m_1} & \overbrace{\begin{matrix} x & x \\ x & x \\ x & x \end{matrix}}^{m_2} & \overbrace{\begin{matrix} x & x \\ x & x \\ 0 & 0 \\ & 0 \end{matrix}}^{m_3} \\
 \hline
 & \begin{matrix} 0 & 0 \\ & 0 \end{matrix} & \begin{matrix} y & y \\ y & y \end{matrix} \\
 \hline
 & & \begin{matrix} 0 & 0 \\ & 0 \end{matrix}
 \end{array}$$

Weyr:  $(3, 2, 2)$

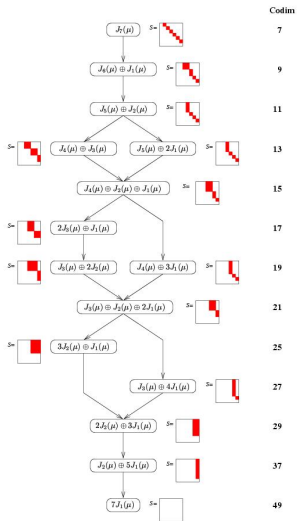
$$\begin{aligned}
 m_1 &= 3 = \dim \mathcal{N}(A - \mu I), \\
 m_1 + m_2 &= 5 = \dim \mathcal{N}((A - \mu I)^2), \\
 m_1 + m_2 + m_3 &= 7 = \dim \mathcal{N}((A - \mu I)^3)
 \end{aligned}$$

$$J_3(0) \oplus J_3(0) \oplus J_1(0)$$

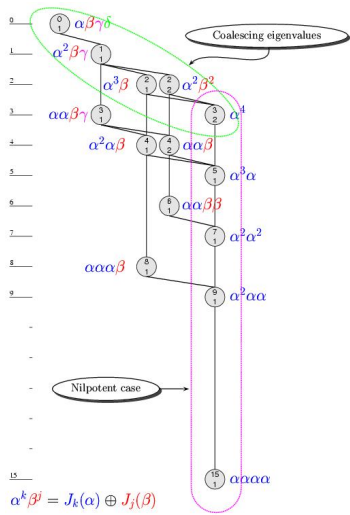
Segre:  $(3, 3, 1)$

# Nilpotent orbit stratification of $7 \times 7$ matrix

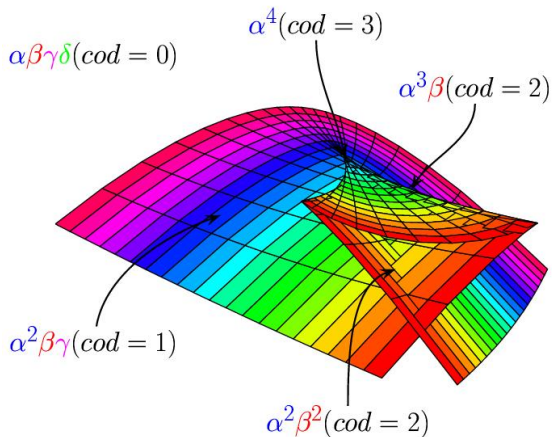
- Dominance ordering of the integer  $n = 7$
- Deformations of stairs and ....
- ... versal deformations
- $A + Z(p)$   
 $\text{span}(Z(p)) = \text{nor}(A)$



# Stratification of $4 \times 4$ matrix bundle



# Swallowtail – bundles of coalescing eigenvalues





A (generalized) state-space system with the *state-space model*

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

can be represented in the form of a *system pencil*

$$\mathbf{S}(\lambda) = G - \lambda H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},$$

with the corresponding general *matrix pencil*  $G - \lambda H$

In short form,  $\mathbf{S}(\lambda)$  is represented by a *matrix quadruple*  $(A, B, C, D)$  ( $E = I$ )

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

- **Controllable system:** There exists an input signal (vector)  $u(t)$ ,  $t_0 \leq t \leq t_f$  that takes every state variable from an initial state  $x(t_0)$  to a desired final state  $x_f$  in finite time.
- **Observable system:** If it possible to find the initial state  $x(t_0)$  from the input signal  $u(t)$  and the output signal  $y(t)$  measured over a finite interval  $t_0 \leq t \leq t_f$ .

Consider the *controllability pair*  $(A, B)$  and the *observability pair*  $(A, C)$ , associated with the particular systems:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$\dot{x}(t) = Ax(t)$$

$$y(t) = Cx(t)$$

System pencil representations:

$$\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I_n \ 0] \quad \text{and} \quad \mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

## An orbit of a matrix pair

A manifold of equivalent matrix pairs:

$$\mathcal{O}(A, B) = \left\{ P(A, B) \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det P \cdot \det Q \neq 0 \right\}$$

## A **bundle** of a matrix pair

The union of all orbits with the same canonical form but with unspecified eigenvalues

$$\mathcal{B}(A, B) = \bigcup_{\mu_i} \mathcal{O}(A, B)$$

A canonical form is the simplest or most symmetrical form a matrix or matrix pencil can be reduced to

- Matrices – [Jordan canonical form](#)
- Matrix pencils – [Kronecker canonical form](#)
- System pencils – [\(generalized\) Brunovsky canonical form](#)

A canonical form reveals the canonical structure information from which the system characteristics are deduced

All matrix pairs in the **same orbit** has the **same canonical form**

# Kronecker canonical form

Any matrix pencil  $G - \lambda H$  or system pencil  $\mathbf{S}(\lambda)$  can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations ( $U$  and  $V$  non-singular):

$$U^{-1}(\mathbf{S}(\lambda))V =$$

$$\text{diag}(L_{\epsilon_1}, \dots, L_{\epsilon_p}, J(\mu_1), \dots, J(\mu_t), N_{s_1}, \dots, N_{s_k}, L_{\eta_1}^T, \dots, L_{\eta_q}^T)$$

Singular part:

- $L_{\epsilon_1}, \dots, L_{\epsilon_p}$  – Right singular blocks
- $L_{\eta_1}^T, \dots, L_{\eta_q}^T$  – Left singular blocks

$$L_k = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}$$

# Kronecker canonical form

Any matrix pencil  $G - \lambda H$  or system pencil  $\mathbf{S}(\lambda)$  can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations ( $U$  and  $V$  non-singular):

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Singular part:

- $L_{\epsilon_1}, \dots, L_{\epsilon_p}$  – Right singular blocks
- $L_{\eta_1}^T, \dots, L_{\eta_q}^T$  – Left singular blocks

$$J_k(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \mu_i - \lambda \end{bmatrix}$$

Regular part:

- $J(\mu_1), \dots, J(\mu_t)$  – Each  $J(\mu_i)$  is block-diagonal with *Jordan* blocks corresponding to the finite eigenvalue  $\mu_i$
- $N_{s_1}, \dots, N_{s_k}$  – Jordan blocks corresponding to the infinite eigenvalue

- $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix}$  has full row rank  $\Rightarrow$  KCF of  $\mathbf{S}_C(\lambda)$  can only have finite eigenvalues (uncontrollable modes) and  $L_k$  blocks:

$$U^{-1}\mathbf{S}_C(\lambda)V = \text{diag}(L_{\epsilon_1}, \dots, L_{\epsilon_p}, J(\mu_1), \dots, J(\mu_t))$$

- $\mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$  has full column rank  $\Rightarrow$  KCF of  $\mathbf{S}_O(\lambda)$  can only have finite eigenvalues (unobservable modes) and  $L_k^T$  blocks:

$$U^{-1}\mathbf{S}_O(\lambda)V = \text{diag}(J(\mu_1), \dots, J(\mu_t), L_{\eta_1}^T, \dots, L_{\eta_p}^T)$$



# Brunovsky canonical form

Given a matrix pair  $(A, B)$  or  $(A, C)$ , there exists a *feedback equivalent* matrix pair in *Brunovsky canonical form* (BCF), such that

$$P \begin{bmatrix} A - \lambda I_n & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} = \left[ \begin{array}{cc|c} A_\epsilon & 0 & B_\epsilon \\ 0 & A_\mu & 0 \end{array} \right]$$

or

$$\begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} P^{-1} = \left[ \begin{array}{cc|c} A_\eta & 0 & \\ 0 & A_\mu & \\ \hline \bar{C}_\eta & & 0 \end{array} \right]$$

respectively, where

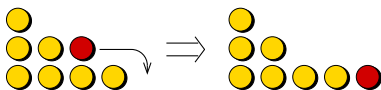
- $(A_\epsilon, B_\epsilon)$  – *controllable* and corresponds to the  $L$  blocks
- $(A_\eta, C_\eta)$  – *observable* and corresponds to the  $L^T$  blocks
- $A_\mu$  – block diagonal with Jordan blocks and corresponds to the *uncontrollable* and *unobservable eigenvalues*, respectively

- $\mathcal{R} = (r_0, r_1, \dots)$  where  $r_i = \#L_k$  blocks with  $k \geq i$
- $\mathcal{L} = (l_0, l_1, \dots)$  where  $l_i = \#L_k^T$  blocks with  $k \geq i$
- $\mathcal{J}_{\mu_i} = (j_1, j_2, \dots)$  where  $j_i = \#J_k(\mu_i)$  blocks with  $k \geq i$ .  
 $\mathcal{J}_{\mu_i}$  is known as the *Weyr characteristics* of the finite eigenvalue  $\mu_i$
- $\mathcal{N} = (n_1, n_2, \dots)$  where  $n_i = \#N_k$  with  $k \geq i$ .  $\mathcal{N}$  is known as the *Weyr characteristics* of the infinite eigenvalue

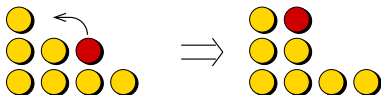
# Integer partitions

A *partition*  $\nu$  of an integer  $K$  is defined as  $\nu = (\nu_1, \nu_2, \dots)$  where  $\nu_1 \geq \nu_2 \geq \dots \geq 0$  and  $K = \nu_1 + \nu_2 + \dots$

*Minimum rightward coin move*: rightward *one* column or downward *one* row (keep partition monotonic)



*Minimum leftward coin move*: leftward *one* column or upward *one* row (keep partition monotonic)



[Edelman, Elmroth & Kågström; 1999]

## Theorem

Given the structure integer partitions  $\mathcal{R}$  and  $\mathcal{J}_{\mu_i}$  of  $(A, B)$ , one of the following if-and-only-if rules finds  $(\tilde{A}, \tilde{B})$  such that:

$\mathcal{O}(A, B)$  **covers**  $\mathcal{O}(\tilde{A}, \tilde{B})$

- 1 Minimum rightward coin move in  $\mathcal{R}$
- 2 If the *rightmost column in  $\mathcal{R}$  is one single coin*, move that coin to a new rightmost column of some  $\mathcal{J}_{\mu_i}$  (which may be empty initially)
- 3 Minimum leftward coin move in any  $\mathcal{J}_{\mu_i}$

Rules 1 and 2: Coin moves that affect  $r_0$  are not allowed

$\mathcal{O}(A, B)$  **is covered by**  $\mathcal{O}(\tilde{A}, \tilde{B})$

- 1 Minimum leftward coin move in  $\mathcal{R}$ , without affecting  $r_0$
- 2 If the *rightmost column in some  $\mathcal{J}_{\mu_i}$  consists of one coin only*, move that coin to a new rightmost column in  $\mathcal{R}$
- 3 Minimum rightward coin move in any  $\mathcal{J}_{\mu_i}$

## Theorem

Given the structure integer partitions  $\mathcal{R}$  and  $\mathcal{J}_{\mu_i}$  of  $(A, B)$ , one of the following if-and-only-if rules finds  $(\tilde{A}, \tilde{B})$  such that:

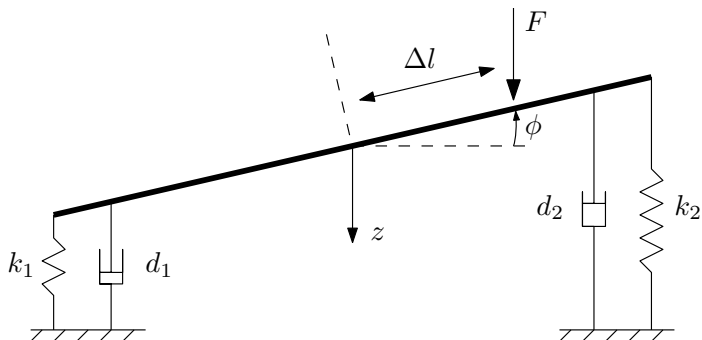
$\mathcal{B}(A, B)$  **covers**  $\mathcal{B}(\tilde{A}, \tilde{B})$

- 1 Minimum rightward coin move in  $\mathcal{R}$
- 2 If the rightmost column in  $\mathcal{R}$  is one single coin, move that coin to the first column of  $\mathcal{J}_{\mu_i}$  for a new eigenvalue  $\mu_i$
- 3 Minimum leftward coin move in any  $\mathcal{J}_{\mu_i}$
- 4 Let any pair of eigenvalues coalesce, i.e., **take the union** of their sets of coins

$\mathcal{B}(A, B)$  **is covered by**  $\mathcal{B}(\tilde{A}, \tilde{B})$

- 1 Minimum leftward coin move in  $\mathcal{R}$ , without affecting  $r_0$
- 2 If some  $\mathcal{J}_{\mu_i}$  consists of one coin only, move that coin to a new rightmost column in  $\mathcal{R}$
- 3 Minimum rightward coin move in any  $\mathcal{J}_{\mu_i}$
- 4 For any  $\mathcal{J}_{\mu_i}$ , **divide the set** of coins into two new sets so that their union is  $\mathcal{J}_{\mu_i}$

## Example 1 – Mechanical system



A **uniform platform** with mass  $m$  and length  $2l$ , supported in both ends by springs

The **control parameter** of the system is the force  $F$  applied at distance  $\Delta l$  from the center of the platform

By linearizing the equations of motion near the equilibrium the system can be expressed by the linear state-space model  $\dot{x} = Ax(\tau) + Bu(\tau)$  [A. Mailybaev '03]:

$$\begin{bmatrix} \omega \dot{z}/l \\ \omega \dot{\phi} \\ \omega^2 \ddot{z}/l \\ \omega^2 \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1 & -c_2 & -f_1 & -f_2 \\ -3c_2 & -3c_1 & -3f_2 & -3f_1 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega \dot{z}/l \\ \omega \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3\Delta \end{bmatrix} \frac{\omega^2}{ml} F$$

where

Fixed elements!

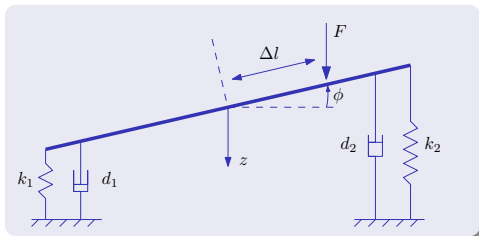
$$c_1 = \frac{(k_1 + k_2)\omega^2}{m}, \quad c_2 = \frac{(k_1 - k_2)\omega^2}{m},$$
$$f_1 = \frac{(d_1 + d_2)\omega}{m}, \quad f_2 = \frac{(d_1 - d_2)\omega}{m},$$

and  $\tau = t/\omega$  where  $\omega$  is a time scale coefficient

# Mechanical system – Canonical forms

With the parameters  $d_1 = 4$ ,  $d_2 = 4$ ,  $k_1 = 6$ ,  $k_2 = 6$ ,  $m = 3$ ,  $l = 1$ ,  $\omega = 0.01$ , and  $\Delta = 0$ , the resulting controllability system pencil  $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$  is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ -0.0004 & 0 & -0.027 & 0 & | & 1 \\ 0 & -0.0012 & 0 & -0.08 & | & 0 \end{bmatrix}$$
$$-\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$





With the parameters  $d_1 = 4$ ,  $d_2 = 4$ ,  $k_1 = 6$ ,  $k_2 = 6$ ,  $m = 3$ ,  $l = 1$ ,  $\omega = 0.01$ , and  $\Delta = 0$ , the resulting controllability system pencil  $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$  is

$$\mathbf{U} \left[ \begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.0004 & 0 & -0.027 & 0 & 1 \\ 0 & -0.0012 & 0 & -0.08 & 0 \end{array} \right] \\
 -\lambda \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \mathbf{V}^{-1}$$

With the parameters  $d_1 = 4$ ,  $d_2 = 4$ ,  $k_1 = 6$ ,  $k_2 = 6$ ,  $m = 3$ ,  $l = 1$ ,  $\omega = 0.01$ , and  $\Delta = 0$ , the resulting controllability system pencil  $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$  is

$$\begin{bmatrix}
 \boxed{0} & \boxed{1} & \boxed{0} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{1} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-0.02} & \boxed{0} & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-0.06} & 0
 \end{bmatrix}$$

$$-\lambda \begin{bmatrix}
 \boxed{1} & \boxed{0} & \boxed{0} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{1} & \boxed{0} & 0 & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & 0 & 0 \\
 \boxed{0} & \boxed{0} & \boxed{0} & 0 & \boxed{1} & 0
 \end{bmatrix}$$

$$= L_2 \oplus J_1(-0.02) \oplus J_1(-0.06)$$

## Kronecker canonical form

With the parameters  $d_1 = 4$ ,  $d_2 = 4$ ,  $k_1 = 6$ ,  $k_2 = 6$ ,  $m = 3$ ,  $l = 1$ ,  $\omega = 0.01$ , and  $\Delta = 0$ , the resulting controllability system pencil  $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$  is

$$\mathbf{P}_{\text{row}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.06 & 0 \end{bmatrix}$$

$$-\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{P}_{\text{col}}$$

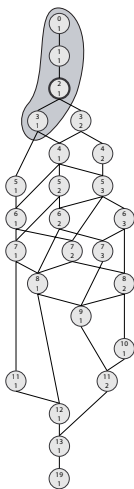
With the parameters  $d_1 = 4$ ,  $d_2 = 4$ ,  $k_1 = 6$ ,  $k_2 = 6$ ,  $m = 3$ ,  $l = 1$ ,  $\omega = 0.01$ , and  $\Delta = 0$ , the resulting controllability system pencil  $\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0]$  is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & -0.02 & 0 & | & 0 \\ 0 & 0 & 0 & -0.06 & | & 0 \end{bmatrix}$$

$$-\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Uncontrollable eigenvalues (modes)

## Brunovsky canonical form

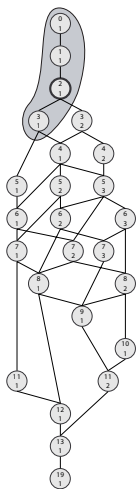


The software tool **StratiGraph** is used for computing and visualizing the stratification

[Elmroth, P. Johansson & Kågström; 2001]

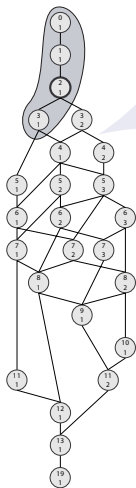
[P. Johansson; PhD Thesis 2006]

# Mechanical system – Illustrating the bundle stratification



Each **node** represents a bundle (or orbit) of a canonical structure

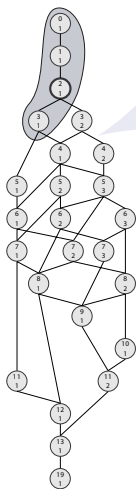
# Mechanical system – Illustrating the bundle stratification



Each **edge** represents a cover relation

It is always possible to go from any canonical structure (node) to another higher up in the graph by a **small perturbation**

# Mechanical system – Illustrating the bundle stratification

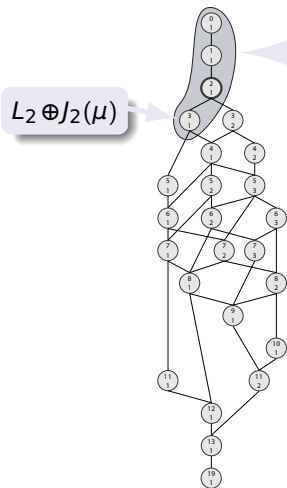


Each **edge** represents a cover relation

A **cover relation** is determined by the **combinatorial rules** acting on the integer sequences representing the canonical structure information



# Mechanical system – Illustrating the bundle stratification



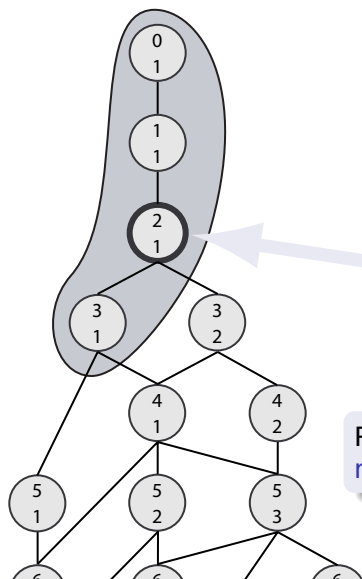
Only possible bundles for the example

## Least generic bundle

Let all **free elements** in the system matrices be zero  $\Rightarrow$  The **least generic possible bundle** has the KCF  $L_2 \oplus J_2(\mu)$

$$\begin{bmatrix} \omega \dot{z}/l \\ \omega \dot{\phi} \\ \omega^2 \ddot{z}/l \\ \omega^2 \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega \dot{z}/l \\ \omega \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{\omega^2}{ml} F$$

# Mechanical system – Illustrating the bundle stratification



$L_2 \oplus$

$\mathcal{R}$ : ●●●

$J_1(-0.02) \oplus$

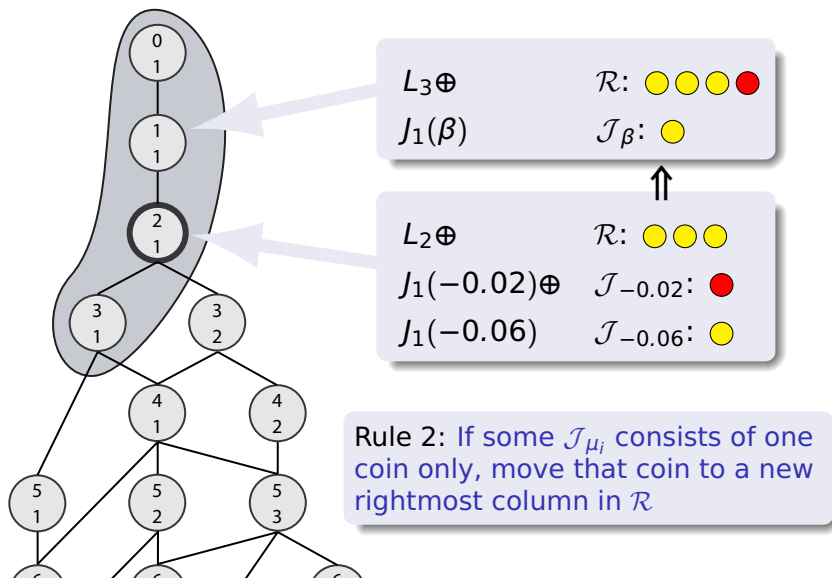
$\mathcal{J}_{-0.02}$ : ●

$J_1(-0.06)$

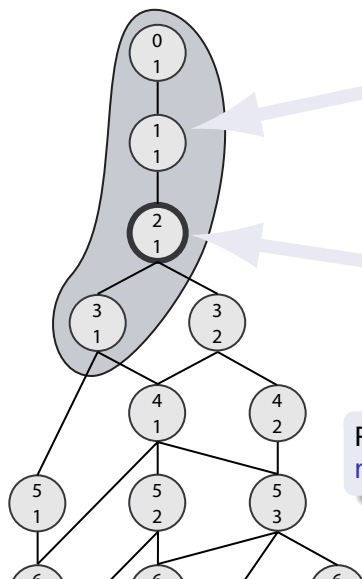
$\mathcal{J}_{-0.06}$ : ●

Rule 1: Minimum leftward coin move in  $\mathcal{R}$ , without affecting  $r_0$

# Mechanical system – Illustrating the bundle stratification



# Mechanical system – Illustrating the bundle stratification



$L_3 \oplus$

$J_1(\beta)$

$L_2 \oplus$

$J_1(-0.02) \oplus$

$J_1(-0.06)$

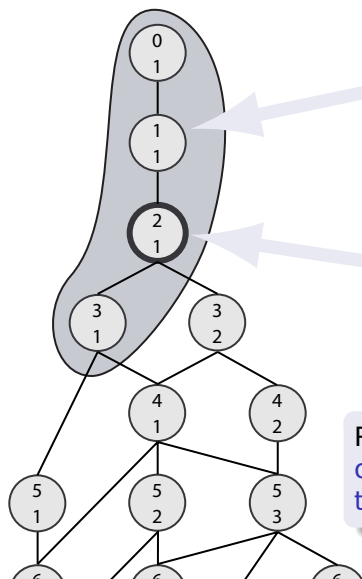
$\mathcal{R}$ : ● ● ●

$\mathcal{J}_{-0.02}$ : ●

$\mathcal{J}_{-0.06}$ : ●

Rule 3: Minimum rightward coin  
move in any  $\mathcal{J}_{\mu_i}$

# Mechanical system – Illustrating the bundle stratification



$L_3 \oplus$

$J_1(\beta)$

$L_2 \oplus$

$\mathcal{R}$ : ● ● ●

$J_1(-0.02) \oplus$

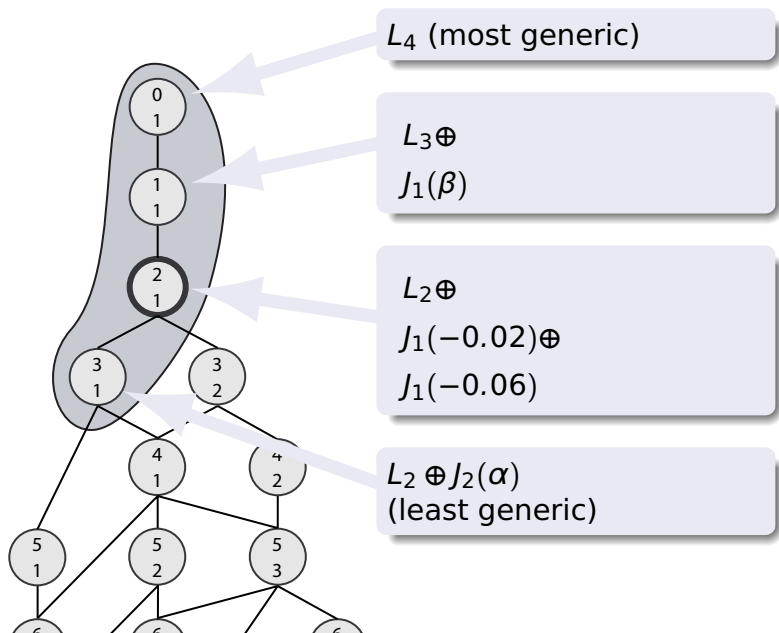
$\mathcal{J}_{-0.02}$ : ●

$J_1(-0.06)$

$\mathcal{J}_{-0.06}$ : ●

Rule 4: For any  $\mathcal{J}_{\mu_i}$ , divide the set of coins into two new sets so that their union is  $\mathcal{J}_{\mu_i}$

# Mechanical system – Illustrating the bundle stratification



## Example 2 – Boeing 747

A Boeing 747 under straight-and-level flight at altitude 600 m with speed 92.6 m/s, flap setting at  $20^\circ$ , and landing gears up. The aircraft has mass = 317,000 kg and the center of gravity coordinates are  $X_{cg} = 25\%$ ,  $Y_{cg} = 0$ , and  $Z_{cg} = 0$



# Boeing 747 – State-space model

A linearized nominal longitudinal model with 5 states and 5 inputs [A. Varga '07]:

$$\dot{x} = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} u(t)$$

$$x = \begin{bmatrix} \delta q \\ \delta V_{TAS} \\ \delta \alpha \\ \delta \theta \\ \delta h_e \end{bmatrix} \begin{pmatrix} \text{pitch rate (rad/s)} \\ \text{true airspeed (m/s)} \\ \text{angle of attack (rad)} \\ \text{pitch angle (rad)} \\ \text{altitude (m)} \end{pmatrix}, \quad u = \begin{bmatrix} \delta e_i \\ \delta e_o \\ \delta i_h \\ \delta EPR_{1,4} \\ \delta EPR_{2,3} \end{bmatrix} \begin{pmatrix} \text{total inner elevator (rad)} \\ \text{total outer elevator (rad)} \\ \text{stabilizer trim angle (rad)} \\ \text{total thrust engine \#1 and \#4 (rad)} \\ \text{total thrust engine \#2 and \#3 (rad)} \end{pmatrix}$$



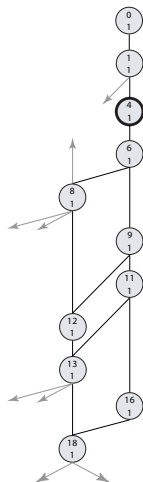
## Goal

Find all *possible* **closest uncontrollable systems** which can be reached by a perturbation of the system matrices, and **distance bounds to uncontrollability**

## Means:

- 1 Identify all the controllable and the nearest uncontrollable systems in the **orbit stratification**
- 2 Determine the **most and least generic orbits** by considering the **structural restrictions** of the system matrices

# Boeing 747 – Illustrating the orbit stratification

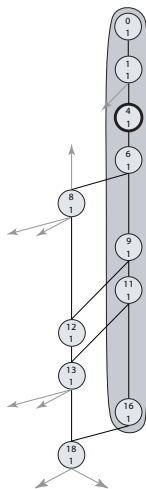


Complete orbit stratification:

- 74 nodes and 133 edges
  - Ranges from codimension 0 to 50
- ⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF  $2L_2 \oplus L_1 \oplus 2L_0$

# Boeing 747 – Illustrating the orbit stratification



Complete orbit stratification:

- 74 nodes and 133 edges
  - Ranges from codimension 0 to 50
- ⇒ Identify only the nodes of interest!

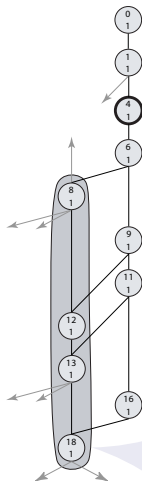
Node corresponding to the orbit of the system under investigation with KCF  $2L_2 \oplus L_1 \oplus 2L_0$

Nodes corresponding to all controllable systems

# Boeing 747 – Illustrating the orbit stratification

Complete orbit stratification:

- 74 nodes and 133 edges
  - Ranges from codimension 0 to 50
- ⇒ Identify only the nodes of interest!



Node corresponding to the orbit of the system under investigation with KCF  $2L_2 \oplus L_1 \oplus 2L_0$

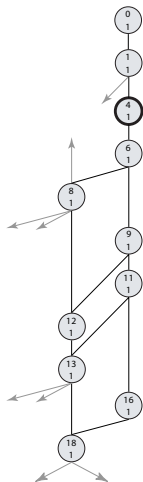
Nodes corresponding to the nearest uncontrollable systems ( $J_1$ -block)

# Boeing 747 – Illustrating the orbit stratification

**NOT POSSIBLE!**

$5L_1$

$L_2 \oplus 3L_1 \oplus L_0$



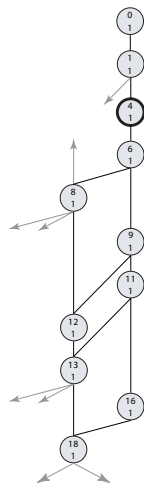
## Most generic orbit

Let all free elements in the system matrices be **nonzero**  $\Rightarrow$

$\#L_0$  blocks =  $5 - \text{rank}(B)$ , i.e., the most generic orbit must have at least two  $L_0$  blocks

$$A = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



## Least generic orbit

Let all **free elements** in the system matrices be **zero**  $\Rightarrow$

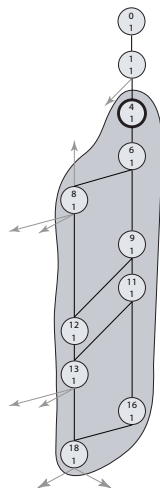
The least generic orbit has the KCF  $5L_0 \oplus J_2(\alpha) \oplus 3J_1(\beta)$  with codimension 42

**No state can be controlled by any of the inputs!**

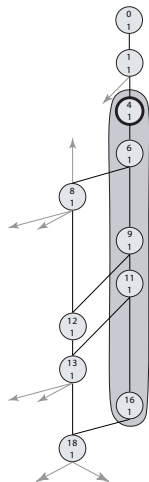
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Boeing 747 – Illustrating the orbit stratification



# Boeing 747 – Controllable orbits



4:  $2L_2 \oplus L_1 \oplus 2L_0$

$\tilde{u}_1$  controls  $\tilde{x}_1, \tilde{x}_2$ ;  $\tilde{u}_2$  controls  $\tilde{x}_3, \tilde{x}_4$ ;  
 $\tilde{u}_3$  controls  $\tilde{x}_5$

6:  $L_3 \oplus 2L_1 \oplus 2L_0$

$\tilde{u}_1$  controls  $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ ;  $\tilde{u}_2$  controls  $\tilde{x}_4$ ;  
 $\tilde{u}_3$  controls  $\tilde{x}_5$

9:  $L_3 \oplus L_2 \oplus 3L_0$

$\tilde{u}_1$  controls  $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ ;  $\tilde{u}_2$  controls  $\tilde{x}_4, \tilde{x}_5$ ;

11:  $L_4 \oplus L_1 \oplus 3L_0$

$\tilde{u}_1$  controls  $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ ;  $\tilde{u}_2$  controls  $\tilde{x}_5$ ;

16:  $L_5 \oplus 4L_0$

$\tilde{u}_1$  controls  $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5$ ;



**Given:**  $m \times n$  pencil  $G - \lambda H$

**Find:** upper and lower bounds on the distance to the closest pencil (say  $K - \lambda L$ ) with a specified KCF

**Upper bound:**

- Find perturbations  $(\delta G, \delta H)$  such that  $(G + \delta G) - \lambda(H + \delta H)$  has the KCF of  $K - \lambda L$
- $(\delta G, \delta H)$  computed by a staircase algorithm that imposes the specified canonical structure (iGUPTRI)
- $\|(\delta G, \delta H)\|_F$  gives the upper bound

## Lower bound:

- Use characterization of tangent space  $\tan(G - \lambda H)$  of the orbit:

$$(XG - GY) - \lambda(XH - HY), \quad \forall X, Y$$

- Now,  $\tan(G - \lambda H)$  is the range of  $T$ , where

$$T \equiv \begin{bmatrix} G^T \otimes I_m & -I_n \otimes G \\ H^T \otimes I_m & -I_n \otimes H \end{bmatrix}$$

- Given  $c = \text{cod}(G - \lambda H)$ , a lower bound to a pencil  $(G + \delta G) - \lambda(H + \delta H)$  with codimension  $c + d$  is

$$\|(\delta G, \delta H)\|_F \geq \frac{1}{\sqrt{m+n}} \left( \sum_{i=2mn-c-d+1}^{2mn} \sigma_i(T) \right)$$

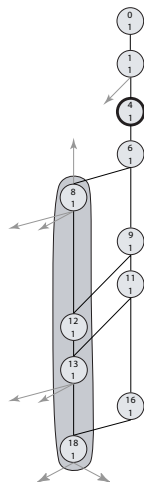
where  $\sigma_i(T) \geq \sigma_{i+1}(T)$

Similar characterizations give lower bounds for matrix pairs with tangent space represented as

$$T_{(A,B)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & I_n \otimes B & 0 \\ B^T \otimes I_n & 0 & I_m \otimes B \end{bmatrix} \quad \text{and}$$

$$T_{(A,C)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & C^T \otimes I_n & 0 \\ -I_n \otimes C & 0 & C^T \otimes I_p \end{bmatrix}$$

Matrix case:  $T_A = I_n \oplus A - A^T \oplus I_n$

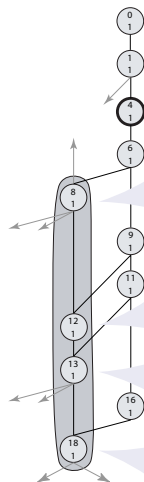


## Distance to uncontrollability

$$\tau(A, B) = \min \left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\| : \right. \\ \left. (A + \Delta A, B + \Delta B) \text{ is uncontrollable} \right\}$$

where  $\| \cdot \|$  denotes the 2-norm or Frobenius norm

# Boeing 747 – Illustrating the orbit stratification



Computed distance to uncontrollability  
[Gu et al., 2006]:  $3.03e-2$

Lower bound:  $4.33e-4$   
Upper bound: 1.0

Lower bound:  $1.09e-3$   
Upper bound:  $2.48e-1$

Lower bound:  $1.33e-3$   
Upper bound:  $1.79e-1$

Lower bound:  $7.57e-2$   
Upper bound:  $5.56e-1$

## Papers

- E. Elmroth, S. Johansson, and B. Kågström  
**Stratification of Controllability and Observability Pairs – Theory and Use in Applications.** *SIAM J. Matrix Analysis and Applications*, Vol. 31, No. 2, 2009
- S. Johansson  
**Reviewing the Closure Hierarchy of Orbits and Bundles of System Pencils and Their Canonical Forms.** Report UMINF-09.02, Umeå University, 2009
- E. Elmroth, P. Johansson, and B. Kågström  
**Bounds for the Distance Between Nearby Jordan and Kronecker Structures in a Closure Hierarchy.** *Journal of Mathematical Sciences*, Vol. 114, No. 6, 2003
- A. Edelman, E. Elmroth, and B. Kågström  
**A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils. Part II: A Stratification-Enhanced Staircase Algorithm.** *SIAM J. Matrix Analysis and Applications*, Vol. 20, No. 3, 1999
- A. Edelman, E. Elmroth, and B. Kågström  
**A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils. Part I: Versal Deformations.** *SIAM J. Matrix Analysis and Applications*, Vol. 18, No. 3, 1997  
(awarded **SIAM/SIAG Linear Algebra Prize 2000**)

## PhD Theses

- S. Johansson  
**Tools for Control System Design – Stratification of Matrix Pairs and Periodic Riccati Differential Equation Solvers.** Department of Computing Science, Umeå University, 2009
- P. Johansson  
**Software Tools for Matrix Canonical Computations and Web-Based Software Library Environments.** Department of Computing Science, Umeå University, 2006

- While **stratigraphy** is the key to understanding the geological evolution of the world, **StratiGraph** is the entry to understanding the "geometrical evolution" of orbits and bundles in the "world" of matrices and matrix pencils.
- But remember these worlds grow exponentially with matrix size!
- Thanks!