Analyzing Controllability and Observability by Exploiting the Stratification of Matrix Pairs

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Based on joint work with
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The Matrix

The Matrix Reloaded

The Matrix Revolutions

The Matrix Stratifications

Coming soon to a PC near you!
THEORY – ALGORITHMS – SOFTWARE TOOLS

**Theme 1:** Matrix Pencil Computations in Computer-Aided Control System Design
- Ill-posed eigenvalue problems
- Canonical forms (Jordan, Kronecker, staircase)
- Generalized Schur forms (GUPTRI, QZ)
- Subspaces: eigenvalue reordering
- Matrix equations (Sylvester, Lyapunov, Riccati)
- Functions of matrices
- Perturbation theory, condition estimation and error bounds
- Periodic counterparts
  - Periodic Riccati differential equations

**Theme 2:** High Performance and Parallel Computing
THEORY – ALGORITHMS – SOFTWARE TOOLS

**Theme 2:** High Performance and Parallel Computing

- Blocking for memory hierarchies (DM, SM, hybrid, multicore, GPGPUs)
  - Explicit (multi level) blocking
  - Recursive blocking
  - Blocked hybrid data structures
- Library software
  - Contributions to LAPACK, ScaLAPACK, SLICOT, ESSL
  - Matrix equations: RECSY and SCASY
- Novel parallel QR algorithm
  - 30 times faster than current ScaLAPACK implementation!
  - Solved 100000 × 100000 dense nonsymmetric eigenvalue problems!
Some motivation and background to stratification of orbits and bundles:
- Canonical forms and structure information
- Matrix and pencil spaces
- Graph representation of a closure hierarchy
- Nilpotent matrix orbit stratification ($7 \times 7$)
- Matrix bundle stratification ($4 \times 4$)

Controllability and observability matrix pairs
- System pencils and equivalence orbits and bundles
- Canonical forms of pairs (Kronecker and Brunovsky)
- Closure and cover relations

Applications in control system design and analysis:
- Mechanical system - uniform platform with 2 springs
- Linearized Boeing 747 model

Stratification [Oxford advanced learner’s dictionary]
The division of something into different layers or groups
Computation of canonical forms (e.g., Jordan, Kronecker, Brunovsky) are ill-posed problems
  - small perturbation of input data may drastically change the computed structure

Compute canonical structure information using so-called staircase algorithms (orthogonal transformations)

Need to provide the user with more information:
  - What other structures are nearby?
  - Upper and lower bounds to other structures

Applications in, e.g., control system design
  - Controllability
  - Observability
Objective: Make use of the geometry of matrix and matrix pencil spaces to solve nearness problems related to Jordan and Kronecker canonical forms

Tools: The theory of stratification of orbits and bundles (and versal deformations)

Our program:
To understand qualitative and quantitative properties of nearby Jordan and Kronecker structures

Deliveries: Interactive tools and algorithms that make these complex theories easily available to end users
Matrix and matrix pencil spaces

- An $n \times n$ matrix can be viewed as a point in $n^2$–dim space
- Numerical computations – move from point to point or manifold to manifold

**Orbit of a matrix**

$$\mathcal{O}(A) = \{ PAP^{-1} : \det P \neq 0 \}$$

Manifold of all matrices with Jordan Normal Form (JNF) of $A$

**Orbit of a pencil**

$$\mathcal{O}(A - \lambda B) = \{ P(A - \lambda B)Q : \det P \det Q \neq 0 \}$$

Manifold of all $m \times n$ pencils in $2mn$–dim space with the Kronecker Canonical Form (KCF) of $A - \lambda B$

- Bundle: $\mathcal{B}(\cdot)$–union of all orbits with the same canonical form but with eigenvalues unspecified
$m \times n$ pencil case

- $\dim(\mathcal{O}(A - \lambda B)) = \dim(\tan(\mathcal{O}(A - \lambda B)))$
- $\text{codim}(\mathcal{O}(A - \lambda B)) = \dim(\text{nor}(\mathcal{O}(A - \lambda B)))$
- $\dim(\mathcal{O}(A - \lambda B)) + \text{codim}(\mathcal{O}(A - \lambda B)) = 2mn$
- $\text{codim}(\mathcal{B}(\cdot)) = \text{codim}(\mathcal{O}(\cdot)) - k$
  $k =$ number of unspecified eigenvalues
Given a matrix and its orbit: What other structures are found within its closure?

**Stratification:**
The closure hierarchy of all possible Jordan structures

We make use of:
- **Graphs** to illustrate stratifications
- **Dominance orderings for integer partitions** in proofs and derivations
The 3D space covers the surface.

The two curves are in the closure of the surface, which is in the closure of the 3D space.

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Closure hierarchy – graph representation

Most generic

3D space

surface

curve 1 curve 2

Least generic (most degenerate)

point
Staircase form of nilpotent $7 \times 7$ matrix

**Weyr:** \((3, 2, 2)\)

**Segre:** \((3, 3, 1)\)

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & x & x \\
0 & x & x \\
0 & x & x \\
0 & 0 & y \\
0 & 0 & y \\
0 & 0 & y \\
0 & 0 & 0 \\
\end{array}
\]

- \(m_1 = 3 = \dim \mathcal{N}(A - \mu I)\),
- \(m_1 + m_2 = 5 = \dim \mathcal{N}((A - \mu I)^2)\),
- \(m_1 + m_2 + m_3 = 7 = \dim \mathcal{N}((A - \mu I)^3)\)

\(J_3(0) \oplus J_3(0) \oplus J_1(0)\)
Nilpotent orbit stratification of $7 \times 7$ matrix

- Dominance ordering of the integer $n = 7$

- Deformations of stairs and ....

- ... versal deformations

- $A + Z(p)$
  $\text{span}(Z(p)) = \text{nor}(A)$
Stratification of $4 \times 4$ matrix bundle

$\alpha^k \beta^j = J_k(\alpha) \oplus J_j(\beta)$
Swallowtail – bundles of coalescing eigenvalues

\[ \alpha \beta \gamma \delta (\text{cod} = 0) \]

\[ \alpha^4 (\text{cod} = 3) \]

\[ \alpha^3 \beta (\text{cod} = 2) \]

\[ \alpha^2 \beta \gamma (\text{cod} = 1) \]

\[ \alpha^2 \beta^2 (\text{cod} = 2) \]
A (generalized) state-space system with the state-space model

\[
E \frac{dx(t)}{dt} = Ax(t) + Bu(t), \\
y(t) = Cx(t) + Du(t),
\]

can be represented in the form of a system pencil

\[
S(\lambda) = G - \lambda H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},
\]

with the corresponding general matrix pencil \( G - \lambda H \)

In short form, \( S(\lambda) \) is represented by a matrix quadruple \( (A, B, C, D) \) \( (E = I) \)
\[ \dot{x}(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t), \]

- **Controllable system:** There exists an input signal (vector) \( u(t) \), \( t_0 \leq t \leq t_f \) that takes every state variable from an initial state \( x(t_0) \) to a desired final state \( x_f \) in finite time.

- **Observable system:** If it possible to find the initial state \( x(t_0) \) from the input signal \( u(t) \) and the output signal \( y(t) \) measured over a finite interval \( t_0 \leq t \leq t_f \).
Consider the *controllability pair* \((A, B)\) and the *observability pair* \((A, C)\), associated with the particular systems:

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

and

\[
\dot{x}(t) = Ax(t) \\
y(t) = Cx(t)
\]

System pencil representations:

\[
S_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix} \quad \text{and} \quad S_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}
\]
An orbit of a matrix pair

A manifold of equivalent matrix pairs:

\[ \mathcal{O}(A, B) = \left\{ P(A, B) \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det P \cdot \det Q \neq 0 \right\} \]

A bundle of a matrix pair

The union of all orbits with the same canonical form but with unspecified eigenvalues

\[ \mathcal{B}(A, B) = \bigcup_{\mu_i} \mathcal{O}(A, B) \]
A canonical form is the simplest or most symmetrical form a matrix or matrix pencil can be reduced to:

- Matrices – Jordan canonical form
- Matrix pencils – Kronecker canonical form
- System pencils – (generalized) Brunovsky canonical form

A canonical form reveals the canonical structure information from which the system characteristics are deduced.

All matrix pairs in the same orbit have the same canonical form.
Kronecker canonical form

Any matrix pencil $G - \lambda H$ or system pencil $\mathbf{S}(\lambda)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations ($U$ and $V$ non-singular):

$$U^{-1}(\mathbf{S}(\lambda))V = \text{diag}(L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, \mathbf{J}(\mu_1), \ldots, \mathbf{J}(\mu_t), N_{s_1}, \ldots, N_{s_k}, L^T_{\eta_1}, \ldots, L^T_{\eta_q})$$

Singular part:
- $L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}$ – Right singular blocks
- $L^T_{\eta_1}, \ldots, L^T_{\eta_q}$ – Left singular blocks

$$L_k = \begin{bmatrix} -\lambda & 1 \\ \vdots & \ddots & \ddots \\ & & -\lambda & 1 \end{bmatrix}$$
Kronecker canonical form

Any matrix pencil $G - \lambda H$ or system pencil $S(\lambda)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations ($U$ and $V$ non-singular):

$$U^{-1}(S(\lambda))V = \text{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t), N_{S_1}, \ldots, N_{S_k}, L_{\eta_1}^T, \ldots, L_{\eta_q}^T)$$

**Singular part:**
- $L_{\epsilon_1}, \ldots, L_{\epsilon_p}$ – Right singular blocks
- $L_{\eta_1}^T, \ldots, L_{\eta_q}^T$ – Left singular blocks

**Regular part:**
- $J(\mu_1), \ldots, J(\mu_t)$ – Each $J(\mu_i)$ is block-diagonal with *Jordan* blocks corresponding to the finite eigenvalue $\mu_i$
- $N_{S_1}, \ldots, N_{S_k}$ – Jordan blocks corresponding to the infinite eigenvalue

$$J_k(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & 1 \\ \mu_i - \lambda & \end{bmatrix}$$
\( \mathbf{S}_c(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix} \) has full row rank \( \Rightarrow \) 
KCF of \( \mathbf{S}_c(\lambda) \) can only have finite eigenvalues (uncontrollable modes) and \( L_k \) blocks:

\[
U^{-1}\mathbf{S}_c(\lambda)V = \text{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t))
\]

\( \mathbf{S}_o(\lambda) = \begin{bmatrix} A & \mathbf{C} \\ \mathbf{C} \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix} \) has full column rank \( \Rightarrow \) 
KCF of \( \mathbf{S}_o(\lambda) \) can only have finite eigenvalues (unobservable modes) and \( L_T^T \) blocks:

\[
U^{-1}\mathbf{S}_o(\lambda)V = \text{diag}(J(\mu_1), \ldots, J(\mu_t), L_{\eta_1}^T, \ldots, L_{\eta_p}^T)
\]
Brunovsky canonical form

Given a matrix pair \((A, B)\) or \((A, C)\), there exists a feedback equivalent matrix pair in Brunovsky canonical form (BCF), such that

\[
P \begin{bmatrix} A - \lambda I_n & B \\ \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} = \begin{bmatrix} A_\epsilon & 0 & B_\epsilon \\ 0 & A_\mu & 0 \end{bmatrix}
\]

or

\[
\begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} P^{-1} = \begin{bmatrix} A_\eta & 0 \\ 0 & A_\mu \end{bmatrix}
\]

respectively, where

- \((A_\epsilon, B_\epsilon)\) – controllable and corresponds to the \(L\) blocks
- \((A_\eta, C_\eta)\) – observable and corresponds to the \(L^T\) blocks
- \(A_\mu\) – block diagonal with Jordan blocks and corresponds to the uncontrollable and unobservable eigenvalues, respectively
\( R = (r_0, r_1, \ldots) \) where \( r_i = \#L_k \) blocks with \( k \geq i \)

\( L = (l_0, l_1, \ldots) \) where \( l_i = \#L^T_k \) blocks with \( k \geq i \)

\( J_{\mu_i} = (j_1, j_2, \ldots) \) where \( j_i = \#J_k(\mu_i) \) blocks with \( k \geq i \). \( J_{\mu_i} \) is known as the Weyr characteristics of the finite eigenvalue \( \mu_i \)

\( N = (n_1, n_2, \ldots) \) where \( n_i = \#N_k \) with \( k \geq i \). \( N \) is known as the Weyr characteristics of the infinite eigenvalue
A partition $\nu$ of an integer $K$ is defined as $\nu = (\nu_1, \nu_2, \ldots)$ where $\nu_1 \geq \nu_2 \geq \cdots \geq 0$ and $K = \nu_1 + \nu_2 + \ldots$.

**Minimum rightward coin move:** rightward one column or downward one row (keep partition monotonic)

Minimum leftward coin move: leftward one column or upward one row (keep partition monotonic)

[Edelman, Elmroth & Kågström; 1999]
Given the structure integer partitions $\mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of $(A, B)$, one of the following if-and-only-if rules finds $(\tilde{A}, \tilde{B})$ such that:

$O(A, B)$ covers $O(\tilde{A}, \tilde{B})$

1. Minimum rightward coin move in $\mathcal{R}$
2. If the rightmost column in $\mathcal{R}$ is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially)
3. Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$

Rules 1 and 2: Coin moves that affect $r_0$ are not allowed

$O(A, B)$ is covered by $O(\tilde{A}, \tilde{B})$

1. Minimum leftward coin move in $\mathcal{R}$, without affecting $r_0$
2. If the rightmost column in some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$
3. Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$
Covering relations for \((A, B)\) bundles

**Theorem**

Given the structure integer partitions \(\mathcal{R}\) and \(\mathcal{J}_{\mu_i}\) of \((A, B)\), one of the following if-and-only-if rules finds \((\tilde{A}, \tilde{B})\) such that:

\[ \mathcal{B}(A, B) \text{ covers } \mathcal{B}(\tilde{A}, \tilde{B}) \]

1. Minimum rightward coin move in \(\mathcal{R}\)
2. If the rightmost column in \(\mathcal{R}\) is one single coin, move that coin to the first column of \(\mathcal{J}_{\mu_i}\) for a new eigenvalue \(\mu_i\)
3. Minimum leftward coin move in any \(\mathcal{J}_{\mu_i}\)
4. Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins

\[ \mathcal{B}(A, B) \text{ is covered by } \mathcal{B}(\tilde{A}, \tilde{B}) \]

1. Minimum leftward coin move in \(\mathcal{R}\), without affecting \(r_0\)
2. If some \(\mathcal{J}_{\mu_i}\) consists of one coin only, move that coin to a new rightmost column in \(\mathcal{R}\)
3. Minimum rightward coin move in any \(\mathcal{J}_{\mu_i}\)
4. For any \(\mathcal{J}_{\mu_i}\), divide the set of coins into two new sets so that their union is \(\mathcal{J}_{\mu_i}\)
A uniform platform with mass $m$ and length $2l$, supported in both ends by springs.

The control parameter of the system is the force $F$ applied at distance $\Delta l$ from the center of the platform.
By linearizing the equations of motion near the equilibrium the system can be expressed by the linear state-space model $\dot{x} = Ax(\tau) + Bu(\tau)$ [A. Mailybaev '03]:

\[
\begin{bmatrix}
\omega \dot{z}/l \\
\omega \phi \\
\omega^2 \dot{z}/l \\
\omega^2 \ddot{\phi}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-c_1 & -c_2 & -f_1 & -f_2 \\
-3c_2 & -3c_1 & -3f_2 & -3f_1
\end{bmatrix}
\begin{bmatrix}
z/l \\
\phi \\
\omega \dot{z}/l \\
\omega \phi
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
1 \\
-3\Delta
\end{bmatrix}
\frac{\omega^2}{ml}F
\]

where

\[
c_1 = \frac{(k_1 + k_2)\omega^2}{m}, \quad c_2 = \frac{(k_1 - k_2)\omega^2}{m},
\]

\[
f_1 = \frac{(d_1 + d_2)\omega}{m}, \quad f_2 = \frac{(d_1 - d_2)\omega}{m},
\]

and $\tau = t/\omega$ where $\omega$ is a time scale coefficient.
With the parameters \( d_1 = 4, \ d_2 = 4, \ k_1 = 6, \ k_2 = 6, \ m = 3, \ l = 1, \ \omega = 0.01 \), and \( \Delta = 0 \), the resulting controllability system pencil \( \mathbf{S}_C(\lambda) = [A \ B] - \lambda [I \ 0] \) is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-0.0004 & 0 & -0.027 & 0 & 1 \\
0 & -0.0012 & 0 & -0.08 & 0
\end{bmatrix}
\]

\[
-\lambda\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $S_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-0.0004 & 0 & -0.027 & 0 & 1 \\
0 & -0.0012 & 0 & -0.08 & 0
\end{pmatrix}
$$

$-\lambda
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} V^{-1}$
With the parameters $d_1 = 4, d_2 = 4, k_1 = 6, k_2 = 6, m = 3, l = 1, \omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $S_C(\lambda) = [A \ B] - \lambda [I \ 0]$ is

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

\[= L_2 \oplus J_1(-0.02) \oplus J_1(-0.06)\]

**Kronecker canonical form**
With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \mathbf{A} \mathbf{B} - \lambda \mathbf{I} [1 \ 0]$ is

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -0.02 & 0 \\
0 & 0 & 0 & 0 & -0.06
\end{bmatrix}
\]

$- \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\mathbf{P}_{\text{col}}$
With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \end{bmatrix}$ is

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.02 & 0 & 0 \\
0 & 0 & 0 & -0.06 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Uncontrollable eigenvalues (modes)

Brunovsky canonical form
The software tool **StratiGraph** is used for computing and visualizing the stratification

[Elmroth, P. Johansson & Kågström; 2001]

[P. Johansson; PhD Thesis 2006]
Each node represents a bundle (or orbit) of a canonical structure.
Each edge represents a cover relation.

It is always possible to go from any canonical structure (node) to another higher up in the graph by a small perturbation.
A cover relation is determined by the combinatorial rules acting on the integer sequences representing the canonical structure information.

Each edge represents a cover relation.
Only possible bundles for the example

Least generic bundle
Let all free elements in the system matrices be zero ⇒ The least generic possible bundle has the KCF $L_2 \oplus J_2(\mu)$

\[
\begin{bmatrix}
\omega \dot{z}/l \\
\omega \dot{\phi} \\
\omega^2 \ddot{z}/l \\
\omega^2 \ddot{\phi}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
z/l \\
\phi \\
\omega \dot{z}/l \\
\omega \dot{\phi}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} \frac{\omega^2}{ml} F
\]
Mechanical system – Illustrating the bundle stratification

Rule 1: Minimum leftward coin move in \( \mathcal{R} \), without affecting \( r_0 \)

\[
\begin{align*}
L_2 & \oplus \\
J_1(-0.02) & \oplus \\
J_1(-0.06) & \oplus
\end{align*}
\]

\( \mathcal{R} \): . . . .

\( \mathcal{I} -0.02\): .

\( \mathcal{I} -0.06\): .
Rule 2: If some $J_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $R$. 

\[ L_3 \oplus \]

\[ J_1(\beta) \]

\[ L_2 \oplus \]

\[ J_1(-0.02) \oplus \]

\[ J_1(-0.06) \]
Rule 3: Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$
Rule 4: For any $\mathcal{J}_{\mu_i}$, divide the set of coins into two new sets so that their union is $\mathcal{J}_{\mu_i}$
Mechanical system – Illustrating the bundle stratification

$L_4$ (most generic)

$L_3 \oplus$

$J_1(\beta)$

$L_2 \oplus$

$J_1(-0.02) \oplus$

$J_1(-0.06)$

$L_2 \oplus J_2(\alpha)$

(least generic)
Example 2 – Boeing 747

A Boeing 747 under straight-and-level flight at altitude 600 m with speed 92.6 m/s, flap setting at 20°, and landing gears up. The aircraft has mass = 317,000 kg and the center of gravity coordinates are $X_{cg} = 25\%$, $Y_{cg} = 0$, and $Z_{cg} = 0$. 
A linearized nominal longitudinal model with 5 states and 5 inputs [A. Varga ’07]:

\[
\dot{x} = \begin{bmatrix}
-0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\
0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\
1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & -92.6 & 92.6 & 0
\end{bmatrix}
\begin{bmatrix} x(t) \end{bmatrix} + 
\begin{bmatrix}
-0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\
0 & 0 & -0.3122 & 0.3998 & 0.3998 \\
-0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u(t) \end{bmatrix}
\]

\[
x = \begin{bmatrix}
\delta q \\
\delta V_{TAS} \\
\delta \alpha \\
\delta \theta \\
\delta h_e
\end{bmatrix}, \quad u = \begin{bmatrix}
\delta_{ei} \\
\delta_{eo} \\
\delta_{ih} \\
\delta EPR_{1,4} \\
\delta EPR_{2,3}
\end{bmatrix}
\]

- pitch rate (rad/s)
- true airspeed (m/s)
- angle of attack (rad)
- pitch angle (rad)
- altitude (m)
- total inner elevator (rad)
- total outer elevator (rad)
- stabilizer trim angle (rad)
- total thrust engine #1 and #4 (rad)
- total thrust engine #2 and #3 (rad)
Goal

Find all possible closest uncontrollable systems which can be reached by a perturbation of the system matrices, and distance bounds to uncontrollability

Means:

1. Identify all the controllable and the nearest uncontrollable systems in the orbit stratification
2. Determine the most and least generic orbits by considering the structural restrictions of the system matrices
Complete orbit stratification:
- 74 nodes and 133 edges
- Ranges from codimension 0 to 50

⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$
Complete orbit stratification:
- 74 nodes and 133 edges
- Ranges from codimension 0 to 50

⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

Nodes corresponding to all controllable systems
Complete orbit stratification:
- 74 nodes and 133 edges
- Ranges from codimension 0 to 50
⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

Nodes corresponding to the nearest uncontrollable systems ($J_1$-block)
Boeing 747 – Illustrating the orbit stratification

Most generic orbit

Let all free elements in the system matrices be nonzero ⇒

\#L_0 \text{ blocks} = 5 - \text{rank}(B), \ i.e., \ the \ most \ generic \ orbit \ must \ have \ at \ least \ two \ L_0 \ blocks

$$A = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -92.6 & 92.6 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
Least generic orbit

Let all free elements in the system matrices be zero \( \Rightarrow \)
The least generic orbit has the KCF
\[ 5L_0 \oplus J_2(\alpha) \oplus 3J_1(\beta) \]
with codimension \( 42 \)

No state can be controlled by any of the inputs!

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Boeing 747 – Illustrating the orbit stratification

Possible orbits of interest
Boeing 747 – Controllable orbits

4: $2L_2 \oplus L_1 \oplus 2L_0$
\[\tilde{u}_1\] controls $\tilde{x}_1, \tilde{x}_2$; \[\tilde{u}_2\] controls $\tilde{x}_3, \tilde{x}_4$; \[\tilde{u}_3\] controls $\tilde{x}_5$

6: $L_3 \oplus 2L_1 \oplus 2L_0$
\[\tilde{u}_1\] controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; \[\tilde{u}_2\] controls $\tilde{x}_4$; \[\tilde{u}_3\] controls $\tilde{x}_5$

9: $L_3 \oplus L_2 \oplus 3L_0$
\[\tilde{u}_1\] controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; \[\tilde{u}_2\] controls $\tilde{x}_4, \tilde{x}_5$

11: $L_4 \oplus L_1 \oplus 3L_0$
\[\tilde{u}_1\] controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$; \[\tilde{u}_2\] controls $\tilde{x}_5$

16: $L_5 \oplus 4L_0$
\[\tilde{u}_1\] controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5$
Distance to nearby structures – upper bounds

Given: \( m \times n \) pencil \( G - \lambda H \)

Find: upper and lower bounds on the distance to the closest pencil (say \( K - \lambda L \)) with a specified KCF

Upper bound:
- Find perturbations \( (\delta G, \delta H) \) such that \( (G + \delta G) - \lambda (H + \delta H) \) has the KCF of \( K - \lambda L \)
- \( (\delta G, \delta H) \) computed by a staircase algorithm that imposes the specified canonical structure (iGUPTRI)
- \( \|(\delta G, \delta H)\|_F \) gives the upper bound
Distance to nearby structures – lower bounds

Lower bound:

- Use characterization of tangent space \( \text{tan}(G - \lambda H) \) of the orbit:

\[
(XG - GY) - \lambda(XH - HY), \quad \forall X, Y
\]

- Now, \( \text{tan}(G - \lambda H) \) is the range of \( T \), where

\[
T \equiv \begin{bmatrix}
G^T \otimes I_m & -I_n \otimes G \\
H^T \otimes I_m & -I_n \otimes H
\end{bmatrix}
\]

- Given \( c = \text{cod} (G - \lambda H) \), a lower bound to a pencil \( (G + \delta G) - \lambda(H + \delta H) \) with codimension \( c + d \) is

\[
\|\delta G, \delta H\|_F \geq \frac{1}{\sqrt{m+n}} \left( \sum_{i=2mn-c-d+1}^{2mn} \sigma_i(T) \right)
\]

where \( \sigma_i(T) \geq \sigma_{i+1}(T) \)
Similar characterizations give lower bounds for matrix pairs with tangent space represented as

\[
T_{(A,B)} = \begin{bmatrix}
A^T \otimes I_n - I_n \otimes A & I_n \otimes B & 0 \\
B^T \otimes I_n & 0 & I_m \otimes B
\end{bmatrix}
\]

and

\[
T_{(A,C)} = \begin{bmatrix}
A^T \otimes I_n - I_n \otimes A & C^T \otimes I_n & 0 \\
-I_n \otimes C & 0 & C^T \otimes I_p
\end{bmatrix}
\]

Matrix case: \(T_A = I_n \oplus A - A^T \oplus I_n\)
Distance to uncontrollability

\[ \tau(A, B) = \min \left\{ \| [\Delta A \ \Delta B] \| : (A + \Delta A, B + \Delta B) \text{ is uncontrollable} \right\} \]

where \( \| \cdot \| \) denotes the 2-norm or Frobenius norm.
Computed distance to uncontrollability
[Gu et al., 2006]: $3.03 \times 10^{-2}$

Lower bound: $4.33 \times 10^{-4}$
Upper bound: $1.0$

Lower bound: $1.09 \times 10^{-3}$
Upper bound: $2.48 \times 10^{-1}$

Lower bound: $1.33 \times 10^{-3}$
Upper bound: $1.79 \times 10^{-1}$

Lower bound: $7.57 \times 10^{-2}$
Upper bound: $5.56 \times 10^{-1}$
References – only some of our work

Papers

- E. Elmroth, S. Johansson, and B. Kågström

- S. Johansson
  *Reviewing the Closure Hierarchy of Orbits and Bundles of System Pencils and Their Canonical Forms*. Report UMINF-09.02, Umeå University, 2009

- E. Elmroth, P. Johansson, and B. Kågström

- A. Edelman, E. Elmroth, and B. Kågström

- A. Edelman, E. Elmroth, and B. Kågström

PhD Theses

- S. Johansson

- P. Johansson
  *Software Tools for Matrix Canonical Computations and Web-Based Software Library Environments*. Department of Computing Science, Umeå University, 2006
While **stratigraphy** is the key to understanding the geological evolution of the world, **StratiGraph** is the entry to understanding the "geometrical evolution" of orbits and bundles in the "world" of matrices and matrix pencils.

But remember these worlds grow exponentially with matrix size!

Thanks!