Covariant Pencils of Matrices

AKA Wavelets

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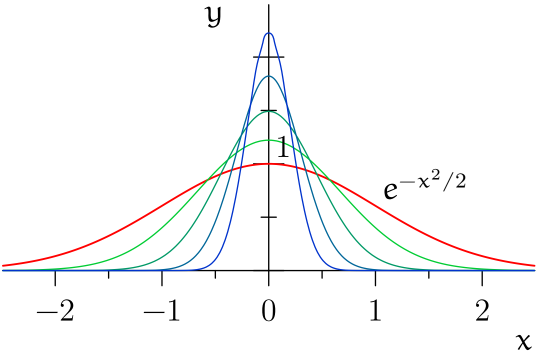
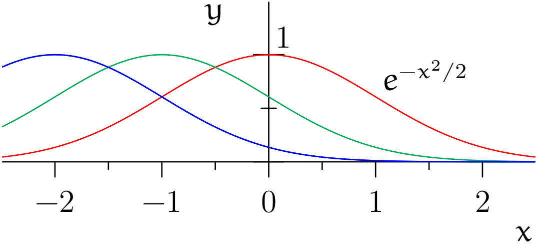
Wavelets from the $ax + b$ group





Scaling and shift (the affine group) create a family of wavelets, which are used for wavelet transform:

$$\begin{aligned}\hat{f}(a,b) &= \left\langle f, \rho_{(a,b)}\phi_0 \right\rangle \\ &= \left\langle f, \phi_{(a,b)} \right\rangle\end{aligned}$$

The Gaussian $\phi(x) = e^{-x^2/2}$ as a mother wavelet produces an approximation of δ-function.

The mother wavelet $\phi(x) = \frac{1}{x+i}$ generates the Cauchy integral formula for the upper half-plane.

Wavelets and Groups

Let G be a group and ρ be its unitary irreducible representation in a Hilbert space H. For a fixed *mother wavelet* $\phi_0 \in H$ define *wavelet transform* from H to $C_b(G)$:

$$[\mathcal{W}f](a,b) = \langle f, \rho_{(a,b)}\phi_0 \rangle = \left\langle \rho_{(a,b)^{-1}} f, \phi_0 \right\rangle .$$

Let Λ be the *left regular representation* on G:

$$\Lambda(g) : f(h) \to f(g^{-1}h).$$

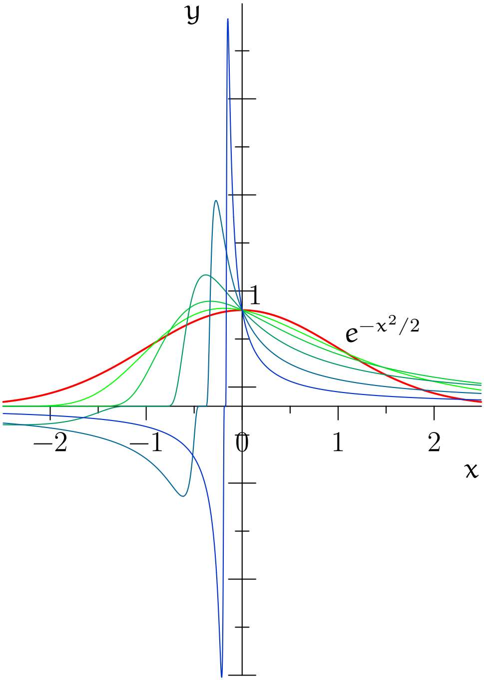
The following properties of the wavelet transform are of interest:

Proposition

1. *$\mathcal{W}$ intertwines ρ and Λ:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

2. *The image of $\mathcal{W}$ is invariant under left shifts.*
3. *The image of $\mathcal{W}$ is generated by images of mother wavelets $\mathcal{W}\phi_0$.*



If we extend the group from $ax + b$ to $SL_2(\mathbb{R})$ then the same Gaussian as the mother wavelet will produces not only an approximation of the δ-function but approximation δ' distribution as well.

However the extension of the group will not affect the mother wavelet $\frac{1}{x+i}$ it will still generate the Cauchy integral, because is an eigenvector of the subgroup K.

Thus choices of group and mother wavelets produce different frameworks.

The group $SL_2(\mathbb{R})$

and operator valued representations

The group $G = SL_2(\mathbb{R})$ consists of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries and the unit determinant $ad - bc = 1$.
Let H be a Hilbert space, for a bounded linear operator T on H we define a paraphrase of the resolvent:

$$R(g) = (cT + dI)^{-1} \qquad \text{where} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

Obviously $R(g)$ contains the essential information about the operator T.
How to extract it?
Consider the space $L(G)$ of functions spanned by all left translations of $R(g)$. The left action $g : f(h) \mapsto f(g^{-1}h)$ of $SL_2(\mathbb{R})$ on this space is a linear representation of this group.

Vector and scalar versions

Function $R(g)$ contain too much information, we may restrict it to get a more detailed view. For vectors $u, v \in H$ we also consider vector and scalar-valued functions related to resolvent:

$$R_u(g) = (cT + dI)^{-1}u, \qquad \text{and} \qquad R_{u,v}(g) = \left\langle (cT + dI)^{-1}u, v \right\rangle$$

where $(cT + dI)^{-1}u$ is understood as a solution w of the equation $u = (cT + dI)w$ if it is unique.

Example

$$T_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad T_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_-(g) = \begin{pmatrix} d & -c \\ c & d \end{pmatrix} \quad R_0(g) = \begin{pmatrix} d & -c \\ -c & d \end{pmatrix} \quad R_+(g) = \begin{pmatrix} d & -c \\ 0 & d \end{pmatrix}$$

Note, $T_\sigma = \sigma I$, that is T_σ is a model for hypercomplex imaginary unit.

Covariant Calculus

Analytic function theory in the upper half-plane $\mathbb{R}^2_+$ is mainly a theory of the *discrete series* representation of $SL_2(\mathbb{R})$ group of 2×2 matrices:

$$\rho_m(g) : f(z) \mapsto \frac{1}{(cz+d)^m} f\left(\frac{az+b}{cz+d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \qquad (1)$$

To get a definition of functional calculus we replace of a *homomorphism property* by a *symmetric covariance*. One possible realisation is:

Definition

An *analytic functional calculus* for an element $T \in \mathfrak{A}$ and an $\mathfrak{A}$-module M is a *continuous linear* mapping $\Phi : A(\mathbb{R}^2_+) \to A(\mathbb{R}^2_+, M)$ such that Φ is an *intertwining operator*

$$\Phi \rho_1 = \rho_T \Phi$$

between two representations of the $SL_2(\mathbb{R})$ group ρ_1 (1) and ρ_T, which is the restriction of the right shifts to the space generated by $R_\nu(g)$.

Covariant Spectrum

as the Support of the Calculus

Definition

A covariant spectrum is defined as the support of covariant functional calculus, that is a decomposition of the intertwining map into primary subrepresentations.

Such components associated with the above $SL_2(\mathbb{R})$ calculus fall into three large classes, with the pairings defined by the usual and indefinite inner products as well as degenerate one.

Example

For the above matrices T_σ the irreducible components are isomorphic to analytic spaces of hypercomplex functions under the fraction-linear transformations:

$$w \mapsto \frac{aw+b}{cw+d}, \qquad \text{where } w = u + \iota v, \text{ and } \iota^2 = -1, 0, 1.$$

$SL_2(\mathbb{R})$ and Its Subgroups

$SL_2(\mathbb{R})$ is the group of 2×2 matrices with real entries and $\det = 1$. A two dimensional subgroup F (F') of lower(upper) triangular matrices:

$$F = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \right\}, \qquad F' = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}, \qquad a \in \mathbb{R}_+,\ b, c \in \mathbb{R}.$$

F is the group of affine transformations of the real line ($ax + b$ group), isomorphic to the upper half-plane.
There are also three one dimensional continuous subgroups:

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix},\ t \in \mathbb{R} \right\}, \tag{2}$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, t \in \mathbb{R} \right\}, \tag{3}$$

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix},\ t \in (-\pi, \pi] \right\}. \tag{4}$$

. . . and Nothing Else

(up to a conjugacy)

Proposition

Any one-parameter subgroup of $SL_2(\mathbb{R})$ is conjugate to either A, N or K.

Proof.

Any one-parameter subgroup is obtained through the exponentiation

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \tag{5}$$

of an element X of the Lie algebra $\mathfrak{sl}_2$ of $SL_2(\mathbb{R})$. Such X is a 2×2 matrix with the zero trace. The behaviour of the Taylor expansion (5) depends from properties of powers X^n. This can be classified by a straightforward calculation. □

Lemma

The square X^2 *of a traceless matrix* $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ *is the identity matrix times* $a^2 + bc = -\det X$. *The factor can be negative, zero or positive, which corresponds to the three different types of the Taylor expansion* (5) *of* e^{tX}.

It is a simple exercise in the Gauss elimination to see that through the matrix similarity we can obtain from X a generator

- of the subgroup K if $(-\det X) < 0$;
- of the subgroup N if $(-\det X) = 0$;
- of the subgroup A if $(-\det X) > 0$.

The determinant is invariant under the similarity, thus these cases are distinct.

$SL_2(\mathbb{R})$ and Homogeneous Spaces

Let G be a group and H be its closed subgroup.
The *homogeneous space* G/H from the equivalence relation: $g' \sim g$ iff $g' = gh$, $h \in H$. The *natural projection* $p : G \to G/H$ puts $g \in G$ into its equivalence class.
A continuous section $s : G/H \to G$ is a right inverse of p, i.e. $p \circ s$ is an identity map on G/H. Then the *left action* of G on itself:

$$\Lambda(g) : g' \mapsto g^{-1} * g', \quad \text{generates} \quad \begin{tikzcd} G \arrow[r, "g*"] \arrow[d, shift left, "p"] & G \arrow[d, shift left, "p"] \\ G/H \arrow[r, "g\cdot"] \arrow[u, shift left, "s"] & G/H \arrow[u, shift left, "s"] \end{tikzcd}$$

If $G = SL_2(\mathbb{R})$ and $H = F$, then $SL_2(\mathbb{R})/F \sim \mathbb{R}$ and $p : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{b}{d}$,:

$$s : u \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad g : u \mapsto p(g^{-1} * s(u)) = \frac{au + b}{cu + d}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$SL_2(\mathbb{R})$ as a Source of Imaginary Units

Consider $G = SL_2(\mathbb{R})$ and H be any subgroup in the Iwasawa decomp:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{6}$$

A right inverse s to the natural projection $p : G \to G/H$:

$$s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2,$$ in the diagram

$$\begin{tikzcd} G \arrow[r, "g*"] \arrow[d, shift left, "p"] & G \arrow[d, shift left, "p"] \\ G/H \arrow[r, "g\cdot"] \arrow[u, shift left, "s"] & G/H \arrow[u, shift left, "s"] \end{tikzcd}$$

defines the G-action $g \cdot x = p(g \cdot s(x))$ on the homogeneous space G/H:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, v) \mapsto \left(\frac{(au + b)(cu + d) - \sigma cav^2}{(cu + d)^2 - \sigma(cv)^2}, \frac{v}{(cu + d)^2 - \sigma(cv)^2} \right).$$

This becomes a Möbius map in (hyper)complex numbers:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw + b}{cw + d}, \qquad w = u + iv, \quad i^2 (:= \sigma) = -1, 0, 1.$$

Möbius Transformations of $\mathbb{R}^2$

For *all* numbers define *Möbius' transformation* of $\mathbb{R}^2 \to \mathbb{R}^2$, (in elliptic and parabolic cases this is even $\mathbb{R}^2_+ \to \mathbb{R}^2_+$!):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ u + \iota v \ \mapsto \ \frac{a(u+\iota v)+b}{c(u+\iota v)+d}. \tag{7}$$

Product $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ gives *Iwasawa* $SL_2(\mathbb{R}) = ANK$. In all $\mathbb{A}$ subgroups A and N acts uniformly:

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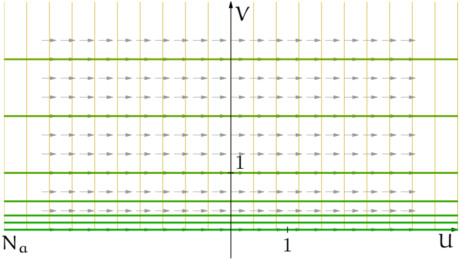

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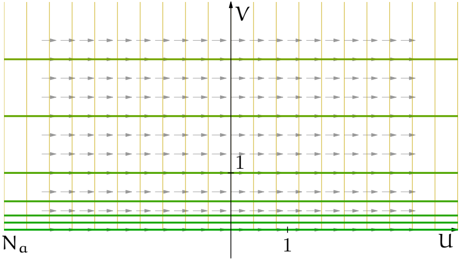
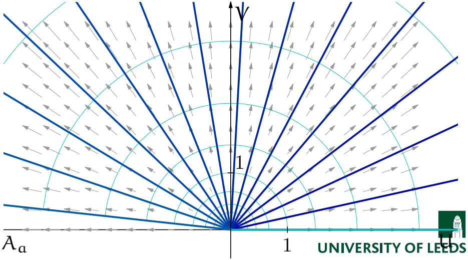
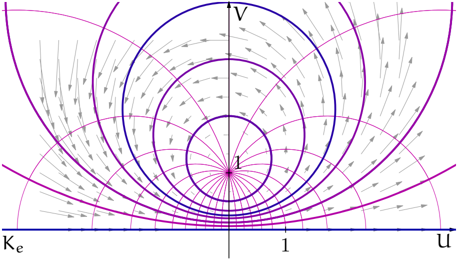


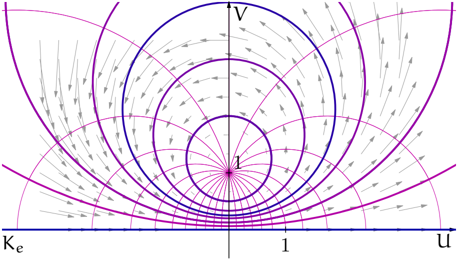


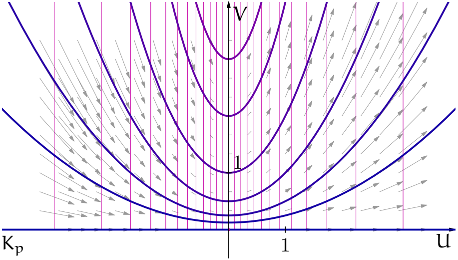


Vector fields are:

$$\begin{aligned} dK_e(u,v) &= (1+u^2-v^2, & 2uv) \\ dK_p(u,v) &= (1+u^2, & 2uv) \\ dK_h(u,v) &= (1+u^2+v^2, & 2uv) \\ dK_\sigma(u,v) &= (1+u^2+\sigma v^2, & 2uv) \end{aligned}$$

Figure: Depending from $i^2=\sigma$ the orbits of subgroup K are circles, parabolas and hyperbolas passing $(0,t)$ with the equation $(u^2-\sigma v^2)+v(\sigma t-t^{-1})+1=0$. This leads to elliptic, parabolic and hyperbolic analytic functions from induced representations.





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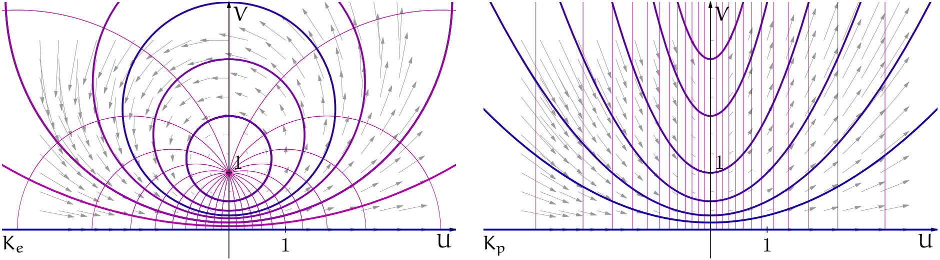

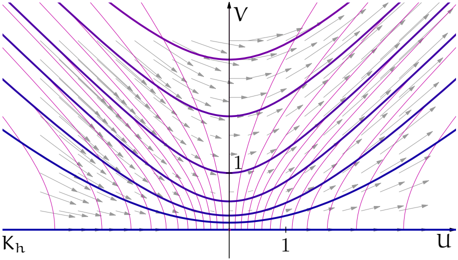

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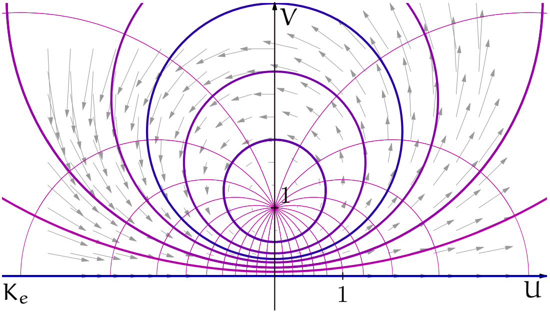
Elliptic, Parabolic, Hyperbolic

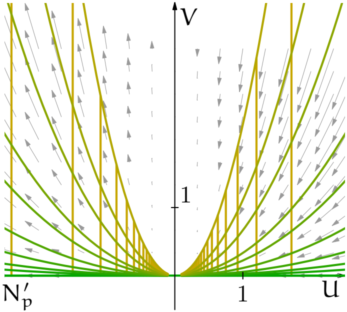
Möbius transformations with the three number systems produce three types of non-Euclidean geometries in the spirit of the Erlangen programme of F. Klein:

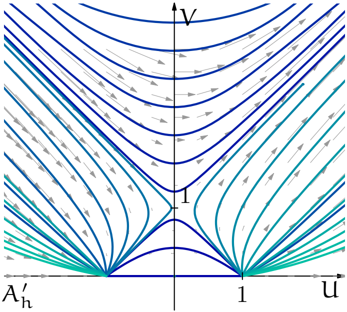
Numbers	complex	dual	double
Imaginary unit	$i^2 = -1$	$\varepsilon^2 = 0$	$\epsilon^2 = 1$
Inv. objects	circles	parabolas	hyperbolas
$\|z\|^2 = z\bar{z}$ is	$x^2 + y^2$	x^2	$x^2 - y^2$
Inv. distance	$\sinh^{-1} \frac{\|z-w\|}{2\sqrt{\Im[z]\Im[w]}}$	$\frac{\|z-w\|}{2\sqrt{\Im[z]\Im[w]}}$	$\sin^{-1} \frac{\|z-w\|}{2\sqrt{\Im[z]\Im[w]}}$
Equidist.	circles	parabolas	hyperbolas
Geodesics	circles	parabolas	hyperbolas

There are many beautiful geometrical results going through all three cases which we will skip in this talk.

Question: can this parallel be extended to analytic functions and operator theory?







Fix subgroups of $(0,1)$ are:

$K = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$—the elliptic case;

$N' = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$—the parabolic case;

$A' = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \exp t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$—the hyperbolic case.

Induced Representations

Let G be a group, H its closed subgroup, χ be a linear representation of H in a space V. The set of V-valued functions with the property

$$F(gh) = \chi(h)F(g),$$

is invariant under left shifts.
The restriction of the left regular representation to this space is called an *induced representation*.
Equivalently we consider the *lifting* of $f(x)$, $x \in X = G/H$ to $F(g)$:

$$F(g) = \chi(h)f(p(g)), \qquad p: G \to X, \quad g = s(x)h, \quad p(s(x)) = x.$$

This is a 1-1 map which transform the left regular representation on G to the following action:

$$[\rho'(g)f](x) = \chi(h)f(g \cdot x), \qquad \text{where} \quad gs(x) = p(gs(x))h.$$

In the case of $SL_2(\mathbb{R})$ we have three different types of actions.

Affine Group

For $G = SL_2(\mathbb{R})$ and $H = F$ the action on G/H is:

$$g : u \mapsto p(g^{-1} * s(u)) = \frac{au+b}{cu+d}, \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We calculate also that

$$r(g^{-1} * s(u)) = \begin{pmatrix} (cu+d)^{-1} & 0 \\ c & cu+d \end{pmatrix}.$$

A generic character of F is a power of its diagonal element:

$$\rho_\kappa \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^\kappa.$$

Thus the corresponding realisation of induced representation is:

$$\rho_\kappa(g) : f(u) \mapsto \frac{1}{(cu+d)^\kappa} f\left(\frac{au+b}{cu+d}\right) \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Induced Wavelet Transform

Let $v_0 \in \mathcal{H}$ be an eigenfunction as follows:

$$\rho(h)v_0 = \chi(h) \cdot v_0, \qquad \text{for all } h \in \tilde{H}.$$

It is suitable to be the *mother wavelet* (*vacuum vector*). Then we have

$$\begin{aligned} [\mathcal{W}f](gh) &= \langle f, \rho(gh)v_0 \rangle = \langle f, \rho(g)\rho(h)v_0 \rangle \\ &= \langle f, \chi(h) \cdot \rho(g)v_0 \rangle = \chi(h) \langle f, \rho(g)v_0 \rangle . \end{aligned}$$

For v_0 the *induced wavelet transform* $\mathcal{W} : \mathcal{H} \to L_\infty(G/H)$ by

$$[\mathcal{W}f](w) = \langle f, \rho_0(s(w))v_0 \rangle , \tag{8}$$

where $w \in G/\tilde{H}$ and $s : G/\tilde{H} \to G$.
It intertwines ρ with a representation induced by χ of $\tilde{H}$.

Cauchy Integral Formula

The infinitesimal version of the eigenvector property $\rho(h)v_0 = \chi(h)\cdot v_0$ is $d\rho_K v_0 = \lambda v_0$, explicitly

$$nuf(u) + f'(u)(1+u^2) = \lambda f(u).$$

The generic solution is:

$$f(u) = \frac{1}{(1+u^2)^{n/2}}\left(\frac{u+i}{u-i}\right)^{i\lambda/2} = \frac{(u+i)^{(i\lambda-n)/2}}{(u-i)^{(i\lambda+n)/2}}.$$

To avoid multivalent function we need to put $\lambda = im$ and the minimal weight condition suggests $m = n$. The induced wavelet transform is:

$$\hat{f}(x,y) = \langle f, \rho_n f_0\rangle = \int_{\mathbb{R}} f(u)\,\frac{\sqrt{y}}{u-x-iy}\,dx = \sqrt{y}\int_{\mathbb{R}} f(u)\,\frac{dx}{u-(x+iy)}$$

The subgroup N consists of shifts, the eigenfunction is $e^{\lambda u}$ and the induced wavelet transform coincides with the Fourier transform.
For the subgroup N' the derived representation is
$d\rho^{N'} = (un)\cdot I - u^2 \cdot \partial_u$.
The corresponding eigenvector is $f_0 = u^n e^{\frac{\lambda}{u}}$.
The induced representation

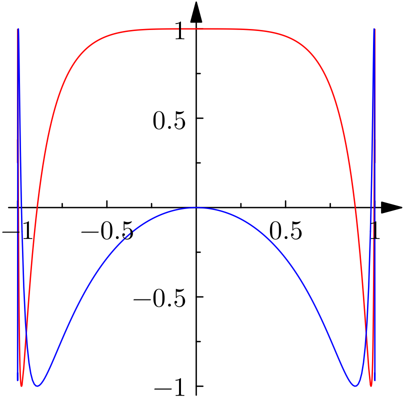
$$\hat{f}(x,y) = \langle f, \rho_n f_0 \rangle \quad = \quad \int_{\mathbb{R}} f(u)\, \frac{(u-x-iy)^n}{\sqrt{y}^n} e^{\frac{\lambda\sqrt{y}}{u-x-iy}}\, dx.$$

(It is a Fourier transform combined with the inversion).

For the subgroup A the derived representation $d\rho_f^{A\,\prime}(u) = -nuf(u) + (u^2-1)f'(u)$. It has two singular point ± 1.

$$f(x) = \frac{1}{(u^2-1)^{n/2}}\left(\frac{u+1}{u-1}\right)^{\lambda/2} = \frac{(u+1)^{(\lambda-n)/2}}{(u-1)^{(\lambda+n)/2}}.$$

The solution has compact support. Its meaning and applicability shall be investigated.



Considering : $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ we chose:

- Eigenvector of K *vacuum vector* $v_0(z) = \dfrac{1}{(x+i)^m}$.
- *wavelets* or *coherent states* $v_m(g,z) = \rho_m(g)v_0(z) = (x-(u+iv))^{-m}$. They depend only from $u+iv \in \mathbb{R}^2_+$.
- The universally defined *wavelet transforms*

$$\mathcal{W}_m : f(z) \mapsto \mathcal{W}_m f(u) = \langle f(z), \rho_m v_0(u,z) \rangle$$

Then $v_m(u,z)$ are the *Cauchy* and *Bergman* kernels. The corresponding wavelet transform became *Cauchy* and *Bergman* integrals:

$$\begin{aligned} \mathcal{W}_1 f(u) &= \frac{1}{2\pi i} \int_{\mathbb{R}} f(z) \frac{1}{u-z}\, dz, \\ \mathcal{W}_m f(u) &= \int_{\mathbb{R}^2_+} f(z) \frac{1}{(\bar{u}-z)^m} (\Im z)^{m-2}\, dz. \end{aligned}$$

Fix Subgroups of i and ϵ

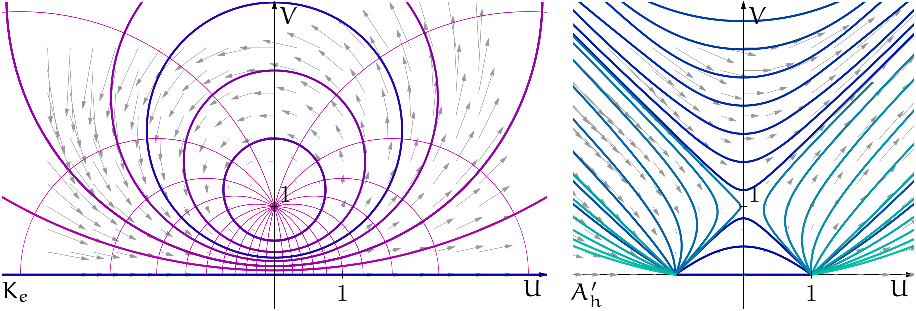


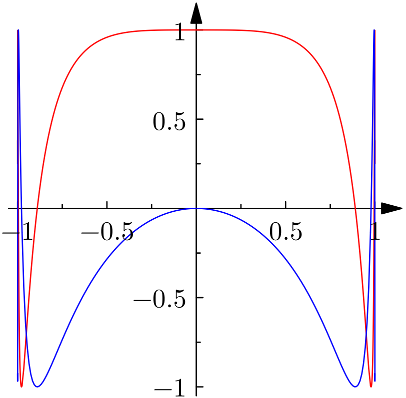
Figure: Elliptic and hyperbolic fix groups of the imaginary units.
In the hyperbolic case there are fixed geometric sets: $\{-1, 1\}$, $(-1, 1)$, $\mathbb{R}$.

Hyperbolic Case in the Half Space

Cauchy Type Integral

In the hyperbolic case we consider *principal* series UIR ρ_σ of $SL_2(\mathbb{R})$. The corresponding decomposition for hyperbolic subgroup A:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = |a| \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$



For the representation ρ_σ an λ-eigenfunction of the vector field A is:

$$f(x) = \frac{1}{(x^2-1)^{\sigma/2}} \left(\frac{x+1}{x-1}\right)^{\lambda/2}.$$

For the value $\lambda = i\sigma$ we have the function:

$$f_0(x) = \frac{1}{(x-\epsilon)^\sigma}.$$

Hyperbolic Wavelets from Double Numbers

The choice of the A-eigenvector as mother wavelet:

- $f_0 = \delta(x \pm 1)$—Dirichlet condition.
- $f_0 = \frac{1}{(x-\epsilon)^\sigma} = \left(\frac{x+\epsilon}{x^2-1}\right)^\sigma$—Neumann condition.
- $f_0 = \frac{\chi(1-x^2)}{(x-\epsilon)^\sigma}$—space-like and time-like separation, Fig. 2.
- ... (combination of above)

Then we follow the general scheme both for wavelets with complex and double valued wavelets:

- *wavelets* or *coherent states* $v_\sigma(g, z) = \rho_\sigma(g) v_0(z)$.
- d'Alambert integral from the universal *wavelet transforms*

$$\mathcal{W}_\sigma : f(z) \mapsto \mathcal{W}_\sigma f(u) = \langle f(z), \rho_\sigma v_0(u, z) \rangle$$

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Hyperbolic Wavelets from Double Numbers

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Parabolic Case in the Half Space

In the parabolic case we consider UIR ρ_π of $SL_2(\mathbb{R})$. The corresponding decomposition for hyperbolic subgroup N':

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = |a| \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

The choice of the N'-eigenvector as mother wavelet:

- $f_0 = \delta(x)$—previous dependence from the single point.
- $f_0 = \dfrac{1}{(x-\varepsilon)^\pi} = \dfrac{1}{x^\pi} + \dfrac{\epsilon}{x^{\pi+1}}$—full Cauchy formula.
- Wavelet generated by the fundamental solution of the heat equation.

Then we follow the general scheme:

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- Heat equation solution from the universal *wavelet transforms*

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Cauchy-Riemann Equation

from Invariant Fields

Let ρ be a unitary representation of Lie group G with the derived representation $d\rho$ of $\mathfrak{g}$. Let a mother wavelet w_0 be a null-solution, i.e. $Aw_0 = 0$, for the operator $A = \sum_J a_j d\rho^{X_j}$, where $X_j \in \mathfrak{g}$. Then the wavelet transform $F(g) = \mathcal{W}f(g) = \langle f, \rho(g) w_0 \rangle$ for any f satisfies to:

$$DF(g) = 0, \qquad \text{where} \quad D = \textstyle\sum_j a_j \mathfrak{L}^{X_j}.$$

Here $\mathfrak{L}^{X_j}$ are left the invariant fields (Lie derivatives) on G corresponding to X_j.

If $\mathfrak{L}^{X_j}$ is derived representation of Lie derivative A, N, K (without the matching subgroup) then C-R operator and Laplacian are given by:

$$D = \iota\mathfrak{L}^{A} + \mathfrak{L}^{X}, \qquad \text{and} \qquad \Delta = D\bar{D} = -\sigma \mathfrak{L}^{A^2} + \mathfrak{L}^{X^2}, \tag{9}$$

where X is in the orthogonal complement (with respect to the Killing form) of the corresponding subgroup K, N, A.

Taylor Expansion over Eigenfunctions

Wavelet are decomposable $v_g(x) = \sum_\alpha c_\alpha(x)\phi_\alpha(g)$ over the *complete set* of its eigenfunctions $\phi_\alpha(u)$ of the principal subgroup. The C-R operators kill half of them ("negative powers"), only the other half is really needed for the decomposition. Then from the wavelet transform:

$$\langle f(z), v_g(z)\rangle = \left\langle f(z), \sum_\alpha c_\alpha(z)\phi_\alpha(g)\right\rangle = \sum_\alpha \phi_\alpha(g)\,\langle f(z), c_\alpha(z)\rangle$$

In the elliptic case eigenvectors of K are z^m, $m = 0, 1, 2, \ldots$ and the decomposition is the *Taylor series*:

$$f(z) = \sum_0^\infty c_n z^n.$$

In the hyperbolic case eigenvectors of A are z^p, $p \in \mathbb{R}_+$ and a Taylor type expansion is given by the integral (not series!)

$$f(\mathsf{z}) = \int_0^\infty c(p)\mathsf{z}^p\,dp.$$

UNIVERSITY OF LEEDS

$SL_2(\mathbb{R})$ Actions on Algebras/Moduli

Let $a \in \mathfrak{A}$ with $\mathbf{sp}\, a \in \bar{\mathbb{D}}$ be fixed in a Banach algebra $\mathfrak{A}$ with the unit e, then

$$g : a \mapsto g \cdot a = (\bar{\alpha} a - \bar{\beta} e)(\alpha e - \beta a)^{-1}, \qquad g \in SL_2(\mathbb{R}) \tag{10}$$

is a well defined $SL_2(\mathbb{R})$ action on a subset $\mathbb{A} = \{g \cdot a \mid g \in SL_2(\mathbb{R})\} \in \mathfrak{A}$, i.e. $\mathbb{A}$ is a $SL_2(\mathbb{R})$-homogeneous space.
Define *resolvent* function $R(g, a) : \mathbb{A} \to \mathfrak{A}$:

$$R(g, a) = (\alpha e - \beta a)^{-1} \quad \text{then} \quad R_1(g_1, \mathsf{a}) R_1(g_2, g_1^{-1}\mathsf{a}) = R_1(g_1 g_2, \mathsf{a}). \tag{11}$$

Thus we can linearise (10) in $C(\mathbb{A}, M)$, for a left $\mathfrak{A}$-module M (e.g. $M = \mathfrak{A}$):

$$\begin{aligned} \rho_a(g_1) : f(g^{-1} \cdot a) \quad &\mapsto \quad R(g_1^{-1} g^{-1}, a) f(g_1^{-1} g^{-1} \cdot a) \\ &= (\alpha' e - \beta' a)^{-1} f\left(\frac{\bar{\alpha}' \cdot a - \bar{\beta}' e}{\alpha' e - \beta' a}\right). \end{aligned}$$

Wavelet Transform in Module Spaces

For any $x \in M'$ define a constant M'-valued *vacuum vector*

$$v_x(a) = x \otimes v_0(g) \in C(\mathbb{A}, M)$$

The *wavelet transform* associated with v_x is defined by the same formula:

$$\mathcal{W}_m f(g) = \langle f, \rho_a(g) v_x \rangle$$

which is vector versions of Cauchy or Bergman integral. It maps $L_2(\mathbb{A})$ to $C(SL_2(\mathbb{R}), \mathbb{C})$.

An integral representation for a covariant calculus Φ is provided by the inverse wavelet transform:

$$\mathcal{M} : f(g) \mapsto f(a, v_0) = \int_G f(g) \rho_a(g)\, dg v_0$$

The *Riesz-Dunford calculus* is given by

$$\Phi : f \mapsto \mathcal{W}_1 f(a) = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \frac{1}{a - z}\, dz$$

where $M = \mathfrak{A}$ and $v_0 = e$.

Jet Spaces

Definition
Two holomorphic functions have nth *order contact* in a point if their value and their first n derivatives agree at that point.
A point $(z, u^{(n)}) = (z, u, u_1, \ldots, u_n)$ of the *jet space* $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having nth contact at the point z.

For a fixed n each holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has nth *prolongation* (or n-*jet*) $jf : \mathbb{D} \to \mathbb{C}^{n+1}$ defined as follows:

$$jf(z) = (f(z), f'(z), \ldots, f^{(n)}(z)).$$

Jet spaces and prolongations were introduced by S. Lie to study symmetries of the differential equations.

Prolongation of Representations

The representation ρ_m of the group $SL_2(\mathbb{R})$ in $B_m(\mathbb{D})$ could be prolonged to a representation $\rho_m^{(n)}$ of $SL_2(\mathbb{R})$ by a transformation $\rho_1^{(n)} : jf(z) \mapsto \rho_1^{(n)} jf(z)$ of the jet space $\mathbb{J}^n$:

$$\rho_m^{(n)}(g) : (z, u, \ldots, u_n) \mapsto (z(g), u(g), \ldots, u_n(g)), \quad \text{where } z(g) = \frac{\bar{\alpha}z - \bar{\beta}}{-\beta z + \alpha},$$

and $u_k(g)$ is the kth derivative of $\rho_m u$ at the point $z(g)$.
From the definition: j intertwines ρ_1 and $\rho_1^{(n)}$:

$$j\rho_1(g) = \rho_1^{(n)}(g) j \quad \text{for all } g \in SL_2(\mathbb{R}).$$

Proposition

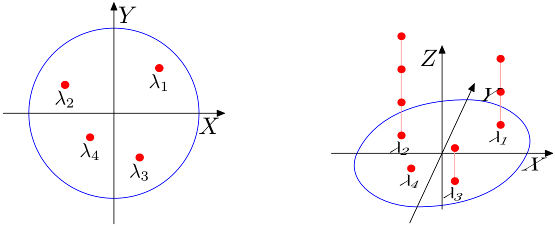
Let a is a Jordan block of a length k for $\lambda = 0$, and x be its root vector of order k, i.e. $a^{k-1}x \neq a^k x = 0$. Then $\rho_{a,m}$ on v_x is equivalent to ρ_m^k.

Spectrum of a Jordan block

Because of the transitive group of inner automorphisms, which could send any $\lambda \in \mathbb{D}$ to 0, we got the complete characterisation of ρ_a for matrices.

Proposition (Jordan normal form)

Representation ρ_a *is equivalent to a direct sum of the prolongations* $\rho_m^{(k)}$ *of* ρ_m *in the* k*th jet space* $\mathbb{J}^k$ *intertwined with inner automorphisms. Consequently the spectrum of* a *(defined via the functional calculus* $\Phi = \mathcal{W}_m$*) consists of exactly* n *pairs* (λ_i, k_i), $\lambda_i \in \mathbb{D}$, $k_i \in \mathbb{Z}_+$, $1 \leqslant i \leqslant n$.



Traditional (left) and covariant (right) spectra of the matrix: $a = J_3(\lambda_1) \oplus J_4(\lambda_2) \oplus J_2(\lambda_3) \oplus J_1(\lambda_4)$.

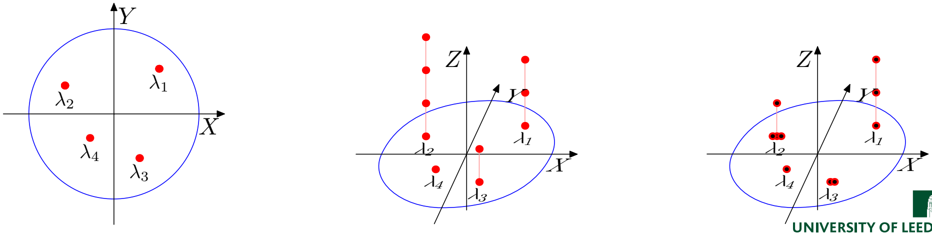
Theorem (Spectral mapping)

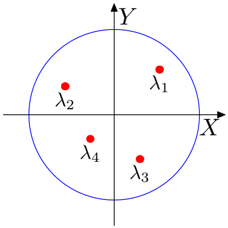
Let $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic map, let us define* $[\phi_* f](z) = f(\phi(z))$ *and its prolongation* $\phi_*^{(n)}$ *onto the jet space* $\mathbb{J}^n$*. Its associated action* $\rho_1^k \phi_*^{(n)} = \phi_*^{(n)} \rho_1^n$ *on the pairs* (λ, k) *is given by the formula:*

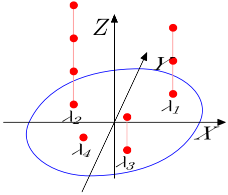
$$\phi_*^{(n)}(\lambda, k) = \left(\phi(\lambda), \left[\frac{k}{\deg_\lambda \phi}\right]\right),$$

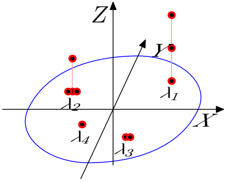
where $\deg_\lambda \phi$ *denotes the degree of zero of the function* $\phi(z) - \phi(\lambda)$ *at the point* $z = \lambda$ *and* $[x]$ *denotes the integer part of* x*. Then*

$\mathbf{sp}\, \phi(a) = \phi_*^{(n)}\, \mathbf{sp}\, a$ (which is actually known for Jordan blocks).









Two first pictures illustrate the traditional and new spectra of the matrix:

$$a = J_3(\lambda_1) \oplus J_4(\lambda_2) \oplus J_2(\lambda_3) \oplus J_1(\lambda_4).$$

The traditional spectrum is *the same* for many essentially different (even by dimensionality) matrices, e.g. for a above and $a_1 = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.
The third picture shows *spectral mapping* for a function ϕ which fixes all four eigenvalues $\lambda_1, \ldots, \lambda_4$ and has such *orders of zeros*: the order 1 at λ_1, exactly the order 3 at λ_2, an order at least 2 at λ_3, and finally any order at λ_4.

Spectrum of a Perturbation of a Jordan Block

Theorem (V.B. Lidskii, 1965)

Eigenvalues of the perturbed matrix $J_n + c^n K$ *admit the expansion*

$$\lambda_j = c\xi^{1/n} + o(c),$$

where $\xi^{1/n}$ *represents all* n*-th roots of certain* $\xi \in \mathbb{C}$.

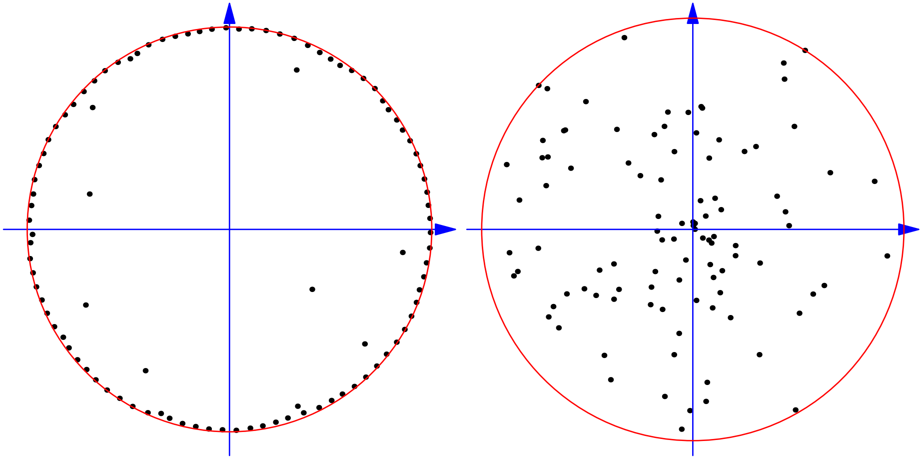
Theorem (E.B. Davies, M. Hager, 2006)

Let $M = J + c^n K$ *where* J *is the standard* $n \times n$ *Jordan matrix,* $0 < c < 1$ *and* K *is a random matrix with independent Gaussian entries.*
Then for any $\varepsilon > 0$ *with probability that converges to* 1 *as* $n \to \infty$*, the proportion of the eigenvalues that lie in any annulus*

$$\{z : c - \varepsilon < |z| < c + \varepsilon\}$$

converges to 1*. The remaining eigenvalues lie inside the annulus.*

Is It "Spectral Instability"?



(a) (b)

Figure: (a) The spectrum of the perturbation $J_{100} + C$ of the Jordan block, see a movie and another one.
(b) The spectrum of the random matrix C

Spectral Distance

The following notion links functional calculus with functional models:

Definition

The spectral distance between two points in the jet space is equal to distance in $H_2(\mathbb{T})$ between two Blaschke products with respective zeroes.

Theorem

Let $n = 2$ and let $\lambda_1(t)$ and $\lambda_2(t)$ are eigenvalues of the matrix $J_2 + c^2 \cdot K$ with some matrix K. Then

$$|\lambda_1(c)| + |\lambda_2(c)| = O(c), \textit{ however } |\lambda_1(c) + \lambda_2(c)| = O(c^2). \qquad (12)$$

The spectral distance from the 1-jet at 0 to two 0-jets at points λ_1 and λ_2 bounded only by the first condition in (12) *is $O(c^2)$.*
However the spectral distance between J_2 and $J_2 + c^2 \cdot K$ is $O(c^4)$.

Hyperbolic Functional Calculus

Elliptic case	**Hyperbolic case**
Variable $z = e^{it} = \cos t + i \sin t$	Variable $z = e^{\epsilon t} = \cosh t + \epsilon \sinh t$
Taylor series $f(z) = \sum_0^\infty c_n z^n$	Taylor integral $f(z) = \int_0^\infty c(p) z^p \, dp$
Multiplication by e^{it}	Multiplication by $e^{\epsilon t}$
Shift S_d on $\mathbb{Z}$	Shift S_c on $\mathbb{R}$
Root vector v_k: $S_d^k v_k = 0$	Root vector v_a: $S_c^a v_a = 0$
Calculus $f(S_d)v = \sum_0^{k-1} c_n S_d^n v_k$	Calculus $f(S_c) = \int_0^a c(t) S_c^t \, dt \, v_a$
$H_n = \mathrm{Lin}\{S_d^n v_k, n = 1, \ldots, k-1\}$	$H_a = \overline{\mathrm{Lin}}\{S_c^t v_a, t \in [0, a)\}$
Jordan block on H_n	Integral operator on H_a

Polynomially Bounded Operators

Standard for a with $\mathbf{sp}\, a \in \bar{\mathbb{D}}$ and $\|a^k\| < Ck^p$ to consider *power bounded* ra, where $0 < r < 1$, and its H_∞ calculus. Further properties of an operator are recovered from consideration of the limit $r \to 1$.

A *better regularisation*, $a^k \to a^k/k^p$ rather than $a^k \to r^k a^k$, is achieved in the covariant calculus framework (although algebra homomorphism is *completely* lost).

Since norm of $f(z) = \sum_0^\infty c_k z^k$ in B_m is equivalent to $\sum_0^\infty c_k^2/k^{m-1}$ for polynomially bounded a the resolvent $R(z, a)$ belongs to any B_m with $m > 2(p+1)$.

Define a representation of $SL_2(\mathbb{R})$ in $B_m(\mathbb{D} \times \mathbb{A}, M)$ by:

$$\rho_m{}' : f(u, a) \mapsto \frac{1}{(\bar{\beta}u + \alpha)^{m-1}(\alpha e - \beta a)} f\left(u, \frac{\bar{\alpha} a - \beta e}{\alpha e - \beta a}\right).$$

It is generated by the discrete series representation of $SL_2(\mathbb{R})$ with the *lowest weight* m.

Calculus of Polynomially Bounded Operators in Bergman Spaces

For the vacuum vector $v_0(u, a) \equiv x$ in $B_m(\mathbb{D} \times \mathbb{A}, M)$, where $(x \in M)$, the corresponding functional calculus is given by the integral:

$$f(g \cdot a) = \int_{\mathbb{D}} \frac{f(u)}{(\beta\bar{u} + \bar{\alpha})^{m-1}(\bar{\alpha}e - \bar{\beta}a)} \frac{du}{(1 - |u|^2)^{m-2}}.$$

For Jordan k-blocks with $|\lambda_i| = 1$ it is equivalent to k-prolongation of $\rho_m{}'$. Since $B_m \subset B_n$ if $m \leqslant n$ the existence ρ_m-calculus implies ρ_n calculus. The minimal m such that B_m covariant calculus exists measures the order of operator.

Several Variables Spectral Theory

For a *joint spectrum* of n-tuple of operators we have many alternatives:

- Weyl functional calculus through the Heisenberg group $\mathbb{H}^n$ acting in $L_2(\mathbb{R}^n)$;
- Segal-Bargmann type functional calculus through the Heisenberg group $\mathbb{H}^n$ acting in $L_2(\mathbb{C}^n)$;
- several complex variables through groups of automorphisms of unit ball or polydisk in $\mathbb{C}^n$. However this is suitable mainly for commuting n-tuples;

Or the Clifford analysis through the Möbius group of conformal maps of $\mathbb{C}^n$. The Clifford algebra $\mathcal{C}\ell(n)$ is spanned by $1, e_1, e_2, \ldots, e_n$ with relations

$$e_k^2 = -1 \qquad \text{and} \qquad e_k e_j = -e_j e_k \quad \text{for } k \neq j.$$

Similarly to complex analysis we could derive a *Cauchy kernel* in $B(H) \otimes \mathcal{C}\ell(n)$ (cf. resolvent):

$$R(A_1, A_2, \ldots, A_n; \lambda_1, \lambda_2, \ldots, \lambda_n) = \left(\sum_1^n e_k A_k - \sum_1^n e_k \lambda_k I \right)^{-1}$$

Joint Spectra of Pauli Matrices

Example

Let $J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the Pauli matrices.

The Cauchy kernel is:

$$\frac{-\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 e_1 e_2}{(\lambda_1^2 + \lambda_2^2)^2} \begin{pmatrix} (-1-\lambda_1)e_1 - \lambda_2 e_2 & e_2 \\ e_2 & (1-\lambda_1)e_1 - \lambda_2 e_2. \end{pmatrix}$$

Comparison of different *joint* spectra for J_1 and J_2:

Clifford spectrum (invertibility of $A - \lambda I$)	$\mathbf{sp}_C(J_1, J_2) = \{(0,0)\}$,
Weyl spectrum (support of operator-valued distribution)	$\mathbf{sp}_W(J_1, J_2) = \mathbb{D}$,
Möbius spectrum (support of the intertwining operator)	$\mathbf{sp}_M(J_1, J_2) = \{\rho_1, \rho_1^{(1)}\}$.

Thank you for your attention!