Resolvents of pencils of Dirichlet-Neumann maps

We consider a cylindrical domain $\Omega \in \mathbb{R}^d$ of the form

$$\Omega = [-1,1] \otimes \Omega',$$

where Ω' is some domain in $\mathbb{R}^{d-1}$. For the application we have in mind, $\Omega' = [-1,1]^{d-1}$ equipped with periodic boundary conditions, so that Ω is a section of a torus and its boundary consists of the end-caps $-1 \otimes \Omega'$ and $+1 \otimes \Omega'$. In this domain we consider a linear elliptic formally self-adjoint second-order PDE

$$Lu = -\nabla.(p(\mathbf{x})\nabla u) + q(\mathbf{x})u = 0.$$

If we equip this PDE with Dirichlet boundary conditions

$$u(-1,\cdot) = u_L, \quad u(+1,\cdot) = u_R,$$

where u_L, u_R are elements of (say) $H^2(\Omega')$, then the Dirichlet to Neumann map associated with L is the 2×2 block operator matrix such that

$$\begin{pmatrix} \partial u/\partial \nu_L \\ \partial u/\partial \nu_R \end{pmatrix} = \begin{pmatrix} N_{LL} & N_{LR} \\ N_{RL} & N_{RR} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}.$$

The Dirichlet to Neumann map is a 1st order pseudo-differential operator. The off-diagonal terms are actually smoothing operators, due to elliptic regularity, at least provided the coefficients and the boundary are sufficiently smooth. Moreover, since the Dirichlet to Neumann map is self-adjoint,

$$N_{RL} = N_{LR}^*, \quad N_{LL} = N_{LL}^*, \quad N_{RR} = N_{RR}^*.$$

Our problem is to show that there is a complex number ζ for which the solution u of the PDE $Lu = 0$ with the given Dirichlet boundary conditions does not satisfy

$$\frac{\partial u}{\partial \nu_R} = -\zeta \frac{\partial u}{\partial \nu_L};; \quad u_R = \zeta u_L.$$

Equivalently, we want to show that the quadratic pencil

$$\zeta^2 N_{LR} + N_{RL} + \zeta(N_{RR} + N_{LL})$$

has a point which is not an eigenvalue.

In the case where the coefficients satisfy certain special symmetries, one has $N_{RL} = N_{LR}$, and then Friedlander proved that the required result holds.

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