Differential operators with λ-dependent boundary conditions

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Eigenproblems for ODE Pencils

$$N(y) = \lambda P(y), \quad y = y(x), \;\; x \in [0,1],$$

$$U_j^0(y) = \lambda U_j^1(y), \quad j = 1, \dots, n.$$

- N is a differential expression of order $n > 0$;
- P is a differential expression of order $p \geq 0$;
- the boundary condition operators should satisfy some (modified) strong Birkhoff regularity hypotheses.

An unbelievably elementary problem (Petterson-König rod)

$$y^{(4)} = \lambda y'',$$

$$y(0) = y'(0) = 0,$$

$$y''(1) = 0, \quad y^{(3)}(1) - \lambda\gamma y'(1) = 0.$$

To write in terms of a linear operator:

- $z := y'', \quad y(x) = (Kz)(x) := \int_0^x (x-t)z(t)dt;$
- K is isomorphism from $L^2(0,1)$ to

$$W^2_{2,BC} := \{y \in W^2_2(0,1) \,|\, y(0) = 0 = y'(0)\}.$$

- Problem becomes

$$z'' = \lambda z, \quad z(1) = 0, \ \ z'(1) = \lambda\gamma \int_0^1 z(t)dt.$$

- $\lambda \int_0^1 z(t)dt = \int_0^1 z''(t)dt = z'(1) - z'(0).$

- Thus z satisfies

$$z'' = \lambda z, \quad z(1) = 0, \quad (1-\gamma)z'(1) + \gamma z'(0) = 0.$$

- The natural operator associated with this problem is not self-adjoint but with real eigenvalues, and has eigenfunctions

$$z_n(x) = \sin(\mu_n(x-1)); \quad \cos(\mu_n) = (1-\gamma^{-1}),$$

  which form a Riesz basis in $L^2(0,1)$;
- eigenfunctions of the original pencil are

$$y_n(x) = (Kz_n)(x) = \frac{1}{\mu_n^2}\{\sin(\mu_n(x-1)) + \sin(\mu_n) + x\mu_n\cos(\mu_n)\}$$

  and form a basis in $W^2_{2,BC}$;

- eigenfunctions $(\mu_n y_n)$ do not form a basis in $W_2^1(0,1)$ or $W^1_{2,BC}$:

$$\|\mu_n y_n\|_1 \sim \left(\frac{3}{2} + \gamma^{-1} + \frac{(1-\gamma^{-1})^2}{3}\right)^{1/2} = O(1)$$

  but if $f(x) = x$ then

$$\langle f, \mu_n y_n \rangle_1 = \frac{4}{3}(1-\gamma^{-1}) + o(1)$$

  which does not tend to zero;
- similarly $(\mu_n^2 y_n)$ do not form a basis in $L^2(0,1)$.

In fact, for this problem, the (y_n) are not even complete in $W_2^1(0,1)$.

Orr-Sommerfeld with λ-dependent boundary conditions

$$\left\{((D^2-\alpha^2))^2 - i\alpha R(u(D^2-\alpha^2)-u'')\right\} y = \lambda(D^2-\alpha^2)y$$

with boundary conditions

$$y(1)=0; \quad y'(1)=0,$$

$$i\alpha R u''(0)y(0) = \lambda(y''(0)+\alpha^2 y(0)),$$

$$y'''(0) - 3\alpha^2 y'(0) - i\alpha R(\gamma(y''(0)+\alpha^2 y(0)) + u'(0)y(0)) = \lambda y'(0).$$

We can reduce this linear pencil problem to a linear operator problem (Shkalikov (1986); M., Shkalikov and Tretter (2003)) and prove the following:

Theorem

The eigen- and associated functions of the pencil problem form a Riesz basis in

$$W^3_{2,BC} = \{y \in W^3_2(0,1) \mid y(1) = 0 = y'(1)\}.$$

Remark

The eigen- and associated functions of the pencil do not form a Riesz basis in $W^2_{2,BC}$ or in $W^1_{2,BC}$ or in $L^2(0,1)$. The difference compared to Petterson-König is that this time, to get rid of λ-dependence in the BCs, we have to work with $z''(0)$, and for this reason we need to be in a Sobolev space one order higher than before.

Sturm-Liouville problems with λ-dependent BCs

$$-\psi'' + q(x)\psi = \lambda\psi, \quad x \in [0,1], \tag{1}$$

$$\psi(0) = 0, \tag{2}$$

$$(\lambda\beta_1 + \alpha_1)\psi(1) = (\lambda\beta_2 + \alpha_2)\psi'(1), \tag{3}$$

in which

$$\rho := \det \begin{pmatrix} \beta_1 & \alpha_1 \\ \beta_2 & \alpha_2 \end{pmatrix} > 0.$$

This can be formulated as a self-adjoint problem in $L^2(0,1) \oplus \mathbb{C}$ (Friedman, 1956; J. Walter, 1973) with inner product

$$\langle \underline{\psi}, \underline{\phi} \rangle = \rho^{-1} c_\psi \overline{c_\phi} + \int_0^1 \psi \overline{\phi}; \qquad \underline{\psi} = \begin{pmatrix} \psi \\ c_\psi \end{pmatrix}.$$

The eigenfunctions of L form an ONB in the extended space, and their first components form a Riesz basis in $L^2(0,1)$ [J. Walter, 1973; Fulton, 1976].

Compact Potential Resonance Problems in QM

Find $k \in \mathbb{C}$ such that

$$-\psi'' + q(x)\psi = k^2\psi, \quad x \in [0, 1], \tag{4}$$

has a non-trivial solution with

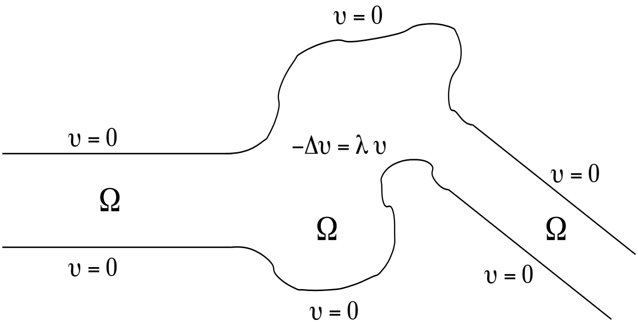
$$\psi(0) = 0; \quad \psi'(1) = ik\psi(1). \tag{5}$$

This problem is a k-quadratic pencil problem with a linearly k-dependent boundary condition. It has some particularly nasty properties:

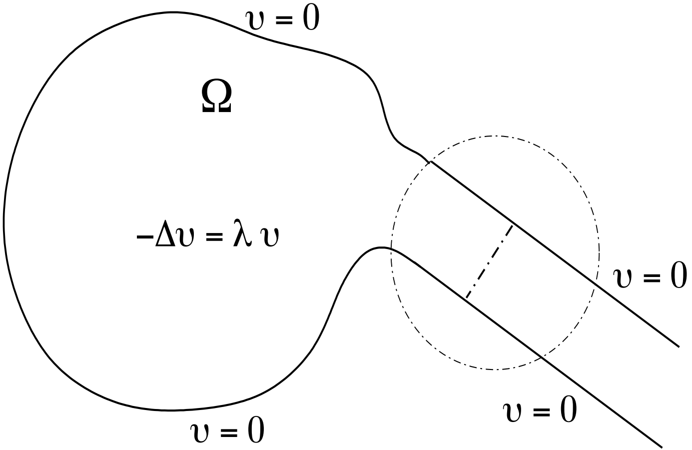
> *The eigen- and associated functions do not form a basis in $L^2(0, 1)$ and may not even be complete.*

Waveguides: PDEs on tubes and Glazman decomposition

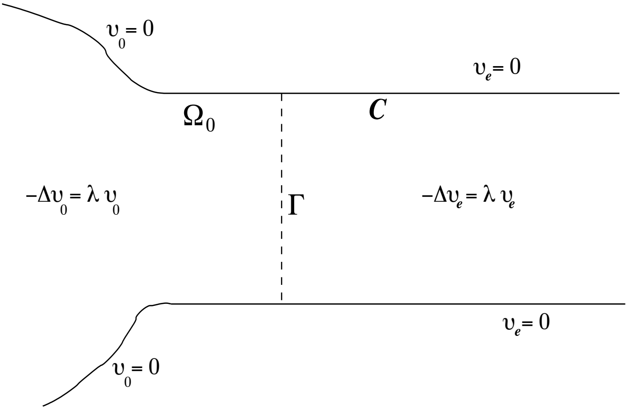
The simplest such problem is that of finding eigenvalues for the Laplacian in a domain with (infinite) cylindrical ends:



Glazman decomposition:



Zoom in on one of the interfaces:



On interface match exterior solution v_e with interior solution v_0; also match normal derivatives:

$$v_e\big|_\Gamma = v_0\big|_\Gamma\,; \quad \frac{\partial v_e}{\partial \nu_e} = -\frac{\partial v_0}{\partial \nu_0}.$$

We define

$\mathcal{R}_e(\lambda)$: exterior Neumann to Dirichlet map on Γ,

$\mathcal{R}_0(\lambda)$: interior Neumann to Dirichlet map on Γ.

Then

$$v_e|_\Gamma = \mathcal{R}_e(\lambda)\frac{\partial v_e}{\partial \nu_e}$$

$$v_0|_\Gamma = \mathcal{R}_0(\lambda)\frac{\partial v_0}{\partial \nu_0}$$

and the matching condition becomes:

$$\ker(\mathcal{R}_e(\lambda) + \mathcal{R}_0(\lambda)) \neq \{0\}.$$

Equivalently,

$\sigma = -1$ is an eigenvalue of the pencil $\sigma\mathcal{R}_e(\lambda) - \mathcal{R}_0(\lambda)$.

Questions:

- For which λ are the operators $\mathcal{R}_e(\lambda)$ and $\mathcal{R}_0(\lambda)$ well defined?
- How do we represent these operators for practical purposes?
- How do the pencil eigenvalues σ depend on λ?

Interior Neumann-Dirichlet map

- $\mathcal{R}_0(\lambda)$ is a meromorphic function of λ;
- the residues at the poles are all negative self-adjoint operators;
- For $\lambda \in \mathbb{R}$ between Neumann eigenvalues, $\mathcal{R}_0(\lambda)$ is an increasing function of λ.

In order to represent $\mathcal{R}_0(\lambda)$ we use the following:

- an orthonormal basis on interface: $(\phi_k)_{k=1}^{\infty}$ spanning $L^2(\Gamma)$;
- interior Neumann eigenvalues $(\mu_m)_{m=1}^{\infty}$ and eigenfunctions $(U_m)_{m=1}^{\infty}$.

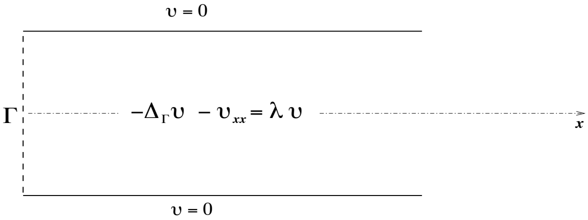
An elementary calculation gives the **Mittag-Leffler expansion**

$$\langle \mathcal{R}_0(\lambda)\phi_k, \phi_\ell \rangle_\Gamma = \sum_{m=1}^{\infty} \frac{1}{\mu_m - \lambda} \langle \phi_k, U_m|_\Gamma \rangle_\Gamma \cdot \langle U_m|_\Gamma, \phi_\ell \rangle_\Gamma \ .$$

This expression can be used to prove the properties of $\mathcal{R}_0(\lambda)$ listed above.

Exterior Neumann-Dirichlet map

Decompose $\mathbf{x} = (x, \mathbf{y})$ where $\mathbf{y}$ is transverse coordinate in Γ:



Introducing the eigenfunctions $(w_n)_{n=1}^{\infty}$ on the cross-section Γ,

$$-\Delta_\Gamma w_n = \kappa_n w_n, \quad n = 1, 2, \dots,$$

$$\mathcal{R}_e(\lambda) g = \sum_{n=1}^{\infty} \frac{\langle g, w_n \rangle_\Gamma}{\sqrt{\kappa_n - \lambda}} w_n(\mathbf{y}).$$

This allows the matrix elements $\langle \mathcal{R}_e(\lambda)\phi_j, \phi_k \rangle_\Gamma$ to be calculated:

$$\langle \mathcal{R}_e(\lambda)\phi_j, \phi_k \rangle_\Gamma = \sum_{n=1}^{\infty} \frac{\langle \phi_j, w_n \rangle_\Gamma \langle w_n, \phi_k \rangle_\Gamma}{\sqrt{\kappa_n - \lambda}}.$$

Behaviour of eigenvalues $\sigma(\lambda)$ of $\sigma\mathcal{R}_e(\lambda) - \mathcal{R}_0(\lambda)$

The pencil eigenvalues are monotone increasing between interior Neumann eigenvalues:

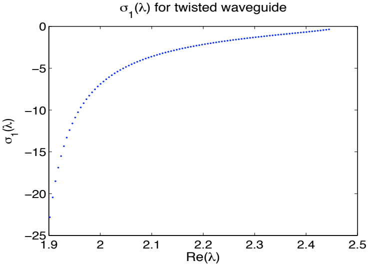


Figure: Pencil eigenvalue as a function of λ

- Monotonicity makes it easy in principle to find λ s.t. $\sigma_j(\lambda) = -1$ for some j...
- ...but isn't every evaluation of $\sigma_j(\lambda)$ very expensive?

Efficiency issues

In the expressions

$$\langle \mathcal{R}_0(\lambda)\phi_k, \phi_j\rangle_\Gamma = \sum_{m=1}^{\infty} \frac{1}{\mu_m - \lambda} \langle \phi_k, U_m|_\Gamma\rangle_\Gamma \cdot \langle U_m|_\Gamma, \phi_\ell\rangle_\Gamma \ ,$$

$$\langle \mathcal{R}_e(\lambda)\phi_j, \phi_k\rangle_\Gamma = \sum_{n=1}^{\infty} \frac{\langle \phi_j, w_n\rangle_\Gamma \langle w_n, \phi_k\rangle_\Gamma}{\sqrt{\kappa_n - \lambda}} \ ,$$

the most expensive parts can be calculated independently of λ at the outset.

The speed of convergence of the infinite series can be substantially accelerated by calculating differences, e.g.

$$\langle \mathcal{R}_0(\lambda)\phi_k, \phi_j\rangle_\Gamma - \langle \mathcal{R}_0(\lambda_{ref})\phi_k, \phi_j\rangle_\Gamma$$

and computing the Neumann-Dirichlet matrix elements $\langle \mathcal{R}_0(\lambda_{ref})\phi_k, \phi_j\rangle_\Gamma$ by solving boundary value problems in the usual way. (What is the reason in terms of PDEs?)

Example: bent waveguide in $\mathbb{R}^2$

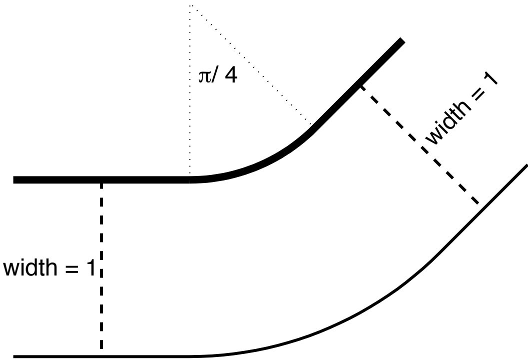


Figure: Twisted Waveguide

Essential spectrum is known:

$$\sigma_{\text{ess}} = [\pi^2/4, +\infty).$$

Exner et al. show that bending the waveguide in the direction of the Dirichlet boundary condition will cause an eigenvalue to appear below the essential spectrum.

Accuracy	Eigenvalue found
1 mesh refinement; sum over $\mu_m \leq 10$	none found
3 mesh refinements; sum over $\mu_m \leq 50$	2.3461
4 mesh refinements; sum over $\mu_m \leq 100$	2.3459
5 mesh refinements; sum over $\mu_m \leq 200$	2.3454

Table: Levels of accuracy and eigenvalue found below the essential spectrum, caused by bending the waveguide.

Resonant waveguides: almost-trapping of waves

This problem is considered in detail by Aslanyan, Parnovski and Vassiliev (2000).

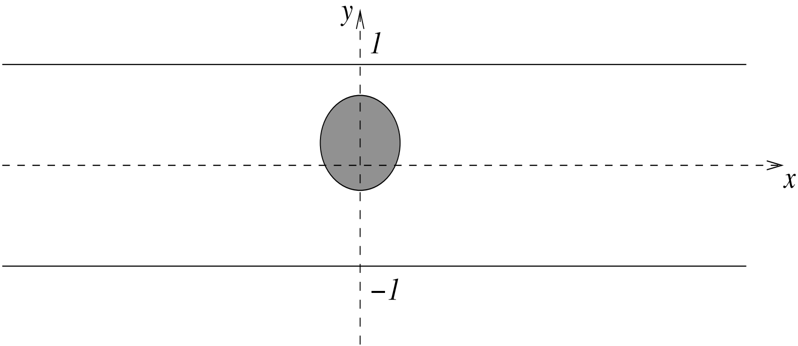


Figure: Waveguide obstructed by a symmetric obstacle centred at $(0, \delta)$.

δ	λ
0.0	1.50499 (low) 1.50486 (high) 1.5048 (A. Aslanyan)
0.1	$1.5080 + 10^{-4}i$ (low) $1.5078 + 10^{-4}i$ (high) $1.5102 + 10^{-4}i$ (AA)
0.2	$1.5167 + 5 \times 10^{-4}i$ (low) $1.5165 + 5 \times 10^{-4}i$ (high) $1.5188 + 5 \times 10^{-4}i$ (AA)

Table: Experiments on the obstructed waveguide.

Can we reduce our problem to a polynomial-in-λ pencil?

Or: what if the PDE is more complicated and we cannot compute $\mathcal{R}_e$?

We want to find λ such that

$$0 \in \operatorname{Spec}_p \left(\mathcal{R}_e(\lambda) + \mathcal{R}_0(\lambda)\right).$$

- We can truncate the Mittag-Leffler expansion of $\mathcal{R}_0(\lambda)$...
- ...but $\mathcal{R}_e(\lambda)$ is harder to approximate without causing **spectral pollution**.

A cheap trick gets round the problem with $\mathcal{R}_e(\lambda)$ in many cases.

1. Choose inner domain Ω_0 'large' so eigenfunctions have decayed well by the time they reach the interface.
2. Replace the PDE with

   $$-\Delta u + i\chi_X(\cdot)u = \lambda u$$

   where, e.g.,

   $$\chi(x, \mathbf{y}) = \begin{cases} 1 & \text{if } |x| < X \\ 0 & \text{if } |x| \geq X \end{cases}$$

   and $X \approx \mathrm{diam}(\Omega_0)/4$ is large. This has the following effect on eigenvalues:

   $$\lambda \mapsto \lambda_R \approx \lambda + i.$$

3. Approximate $\mathcal{R}_e(\cdot)$ very crudely by a constant, e.g. $\mathcal{R}_e \equiv 0$ (Neumann boundary conditions).

It can be proved that this strategy will only pollute exponentially close to the real axis.

δ	λ
0.0	1.5065 1.50486 (previous) 1.5048 (A. Aslanyan)
0.1	1.5075 $1.5078 + 10^{-4}i$ (prev.) $1.5102 + 10^{-4}i$ (AA)
0.2	1.5153 $1.5165 + 5 \times 10^{-4}i$ (prev.) $1.5188 + 5 \times 10^{-4}i$ (AA)

Table: Waveguide revisited