

Variational principles for eigenvalues of operator functions

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Classical variational principle for eigenvalues

\mathcal{H} a Hilbert space

A self-adjoint operator in \mathcal{H} , bounded from below,

i.e. $(Ax, x) \geq c\|x\|^2$ for all $x \in \text{dom}(A)$ and some $c \in \mathbb{R}$.



For the eigenvalues λ_n below the essential spectrum:

$$\lambda_n = \min_{\substack{L \subset \text{dom}(A) \\ \dim L = n}} \max_{\substack{x \in L \\ x \neq 0}} \frac{(Ax, x)}{\|x\|^2} \quad n = 1, 2, \dots$$

$$\lambda_n = \max_{\substack{L \subset \mathcal{H} \\ \dim L = n-1}} \min_{\substack{x \perp L \\ x \in \text{dom}(A), x \neq 0}} \frac{(Ax, x)}{\|x\|^2}$$

[H. Weber, Lord Rayleigh, H. Poincaré, E. Fischer, W. Ritz, ...]

Essential spectrum

If A has N eigenvalues below the essential spectrum, then

$$\inf_{\substack{L \subset \text{dom}(A) \\ \dim L = n}} \max_{\substack{x \in L \\ x \neq 0}} \frac{(Ax, x)}{\|x\|^2} = \min \sigma_{\text{ess}}(A), \quad n > N.$$

Here $\lambda \in \sigma_{\text{ess}}(A)$ if $A - \lambda$ is not Fredholm, i.e.

$$\dim \ker(A - \lambda) = \infty \text{ or } \dim(\mathcal{H} / \text{ran}(A - \lambda)) = \infty.$$

Comparison of two operators

If $\text{dom}(A) \supset \text{dom}(B)$ and

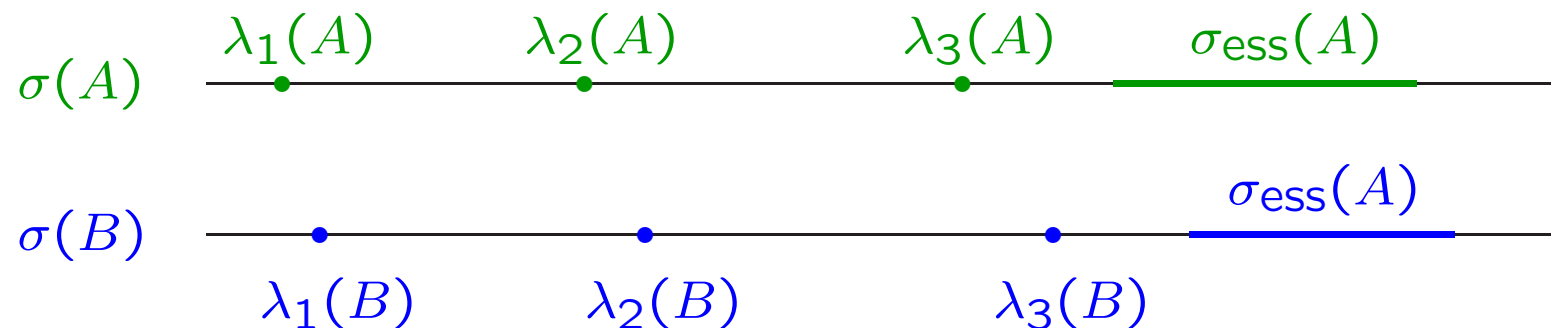
$$(Ax, x) \leq (Bx, x), \quad x \in \text{dom}(B),$$

then

$$\lambda_n(A) \leq \lambda_n(B)$$

and

$$\min \sigma_{\text{ess}}(A) \leq \min \sigma_{\text{ess}}(B).$$



Operator functions and their spectra

$$T: \lambda \mapsto T(\lambda), \quad \lambda \in \Delta$$

where $T(\lambda)$ are closed operators in a Hilbert space \mathcal{H} and $\Delta \subset \mathbb{C}$.

Examples: $T(\lambda) = \lambda^2 A + \lambda B + C$, $T(\lambda) = A - \lambda - B(D - \lambda)^{-1}C$

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λ_0 **eigenvalue** of T $:\iff \exists x_0 \neq 0$, such that $T(\lambda_0)x_0 = 0$

$\lambda_0 \in \sigma(T)$ (**spectrum**) $:\iff T(\lambda_0)$ not boundedly invertible
($\iff 0 \in \sigma(T(\lambda_0))$)

$\lambda_0 \in \sigma_{\text{ess}}(T)$ (**essential spectrum**) $:\iff T(\lambda_0)$ is not **Fredholm**

NB. The usual eigenvalue problem $Ax = \lambda x$ for an **operator** A corresponds to the operator function

$$T(\lambda) = A - \lambda.$$

Generalised Rayleigh functional

Classical Rayleigh functional

$$\frac{(Ax, x)}{(x, x)} \quad \text{for eigenvalue problem } Ax = \lambda x$$

is the **zero** of the function

$$\lambda \quad \mapsto \quad (Ax, x) - \lambda(x, x) = ((A - \lambda)x, x)$$

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For the problem $T(\lambda)x = 0$ take a **zero** of

$$\lambda \mapsto (T(\lambda)x, x)$$

If λ_0 is an eigenvalue of T with eigenvector x_0 , then λ_0 is such a zero for $x = x_0$.

Generalised Rayleigh functional (continued)

Let T be defined on $\Delta \subset \mathbb{R}$ and assume that $T(\lambda)$ is self-adjoint if $\lambda \in \Delta$.

We assume that $(T(\cdot)x, x)$ is continuous and decreasing at value zero for $x \neq 0$, i.e. if $(T(\lambda_0)x, x) = 0$, then

$$(T(\lambda)x, x) > 0 \quad \text{for } \lambda < \lambda_0$$

$$(T(\lambda)x, x) < 0 \quad \text{for } \lambda > \lambda_0$$

Generalised Rayleigh functional (continued)

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Generalised Rayleigh functional:

$$p(x) := \begin{cases} \lambda_0 & \text{if } (T(\lambda_0)x, x) = 0 \\ -\infty & \text{if } (T(\lambda)x, x) < 0 \quad \forall \lambda \in \Delta \\ \infty & \text{if } (T(\lambda)x, x) > 0 \quad \forall \lambda \in \Delta \end{cases}$$

Theorem. [Duffin, Rogers, Werner, Abramov, Markus, ...]

Let T be an operator function defined on some interval $[\alpha, \beta)$ whose values are self-adjoint operators in \mathcal{H} . Assume that

- (i) $\mathcal{D} = \text{dom } T(\lambda)$ is independent of λ
(or $\text{dom } T(\lambda) \subset \mathcal{D} \subset \text{form domain of } T(\lambda)$);
- (ii) T is **continuous** in norm resolvent sense;
- (iii) $(T(\cdot)x, x)$ is **decreasing at value 0** for all $x \in \mathcal{D}$;
- (iv) $T(\alpha) > 0$.

Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of T below the essential spectrum, then

$$\begin{aligned} \lambda_n &= \min_{\substack{L \subset \mathcal{D} \\ \dim L = n}} \max_{\substack{x \in L \\ x \neq 0}} p(x) \\ &= \max_{\substack{L \subset \mathcal{H} \\ \dim L = n-1}} \min_{\substack{x \perp L \\ x \in \mathcal{D}, x \neq 0}} p(x). \end{aligned}$$

Comparison of two operator functions

Let T , \hat{T} be two operator functions as before such that $\mathcal{D} \supset \hat{\mathcal{D}}$ and

$$p(x) \leq \hat{p}(x), \quad x \in \hat{\mathcal{D}}. \quad (*)$$

Then

$$\lambda_n \leq \hat{\lambda}_n.$$

Relation (*) is satisfied if

$$(T(\lambda)x, x) \leq (\hat{T}(\lambda)x, x) \quad \text{for all } \lambda \in [\alpha, \beta), \quad x \in \hat{\mathcal{D}}.$$

Theorem. [Binding, Eschwé, H. Langer 2000; Eschwé, M.L. 2004]
 Let T be an operator function defined on some interval $[\alpha, \beta)$ whose values are self-adjoint operators in \mathcal{H} . Assume that

- (i) $\mathcal{D} = \text{dom}(T(\lambda))$ is independent of λ
 (or $\text{dom } T(\lambda) \subset \mathcal{D} \subset \text{form domain of } T(\lambda)$);
- (ii) T is continuous in norm resolvent sense;
- (iii) $(T(\cdot)x, x)$ is decreasing at value 0 for all $x \in \mathcal{D}$;
- (iv) negative spectrum of $T(\alpha)$ consists of κ eigenvalues.

Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of T below the essential spectrum, then

$$\begin{aligned} \lambda_n &= \min_{\substack{L \subset \mathcal{D} \\ \dim L = n + \kappa}} \max_{\substack{x \in L \\ x \neq 0}} p(x) \\ &= \max_{\substack{L \subset \mathcal{H} \\ \dim L = n - 1 + \kappa}} \min_{\substack{x \perp L \\ x \in \mathcal{D}, x \neq 0}} p(x). \end{aligned}$$

Theorem (continued).

Let T be differentiable and assume in addition that the **Virozub–Matsaev (VM) condition** is satisfied:

$\exists \varepsilon, \delta > 0$ so that for all $x \in \mathcal{D}$, $\|x\| = 1$, and $\lambda \in (\alpha, \beta)$,

$$|(T(\lambda)x, x)| \leq \varepsilon \quad \Rightarrow \quad (T'(\lambda)x, x) \leq -\delta.$$

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If $\sigma_{\text{ess}}(T) \neq \emptyset$ and T has N eigenvalues below $\sigma_{\text{ess}}(T)$, then

$$\inf_{\substack{LCD \\ \dim L = n + \kappa}} \sup_{\substack{x \in L \\ x \neq 0}} p(x) = \min \sigma_{\text{ess}}(T), \quad n > N.$$

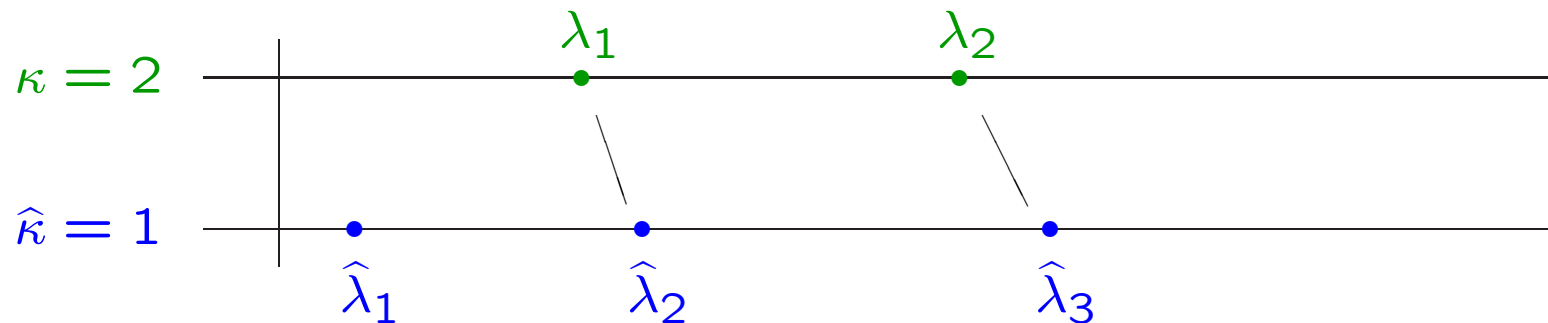
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Then

$$\kappa \geq \hat{\kappa} \quad \text{and} \quad \lambda_{n-\kappa} \leq \hat{\lambda}_{n-\hat{\kappa}}.$$



Klein–Gordon equation

$$\left(- \left(-i\hbar \frac{\partial}{\partial t} - eq \right)^2 + c^2 \left(-i\hbar \nabla - \frac{e}{c} \vec{A} \right)^2 + m^2 c^4 \right) U(x, t) = 0$$

This equation describes particles with spin 0, mass m and charge e .

q ... electrostatic potential

\vec{A} ... electromagnetic potential

c ... speed of light

\vec{A} bounded on \mathbb{R}^3 , $|q(x)| \leq \frac{C}{\|x\|}$, C small enough.

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$$U(x, t) = e^{i\lambda t/\hbar} u(x), \quad A_0 = c^2 (-i\hbar \nabla - \frac{e}{c} \vec{A})^2, \quad \gamma = mc^2, \quad V = eq$$

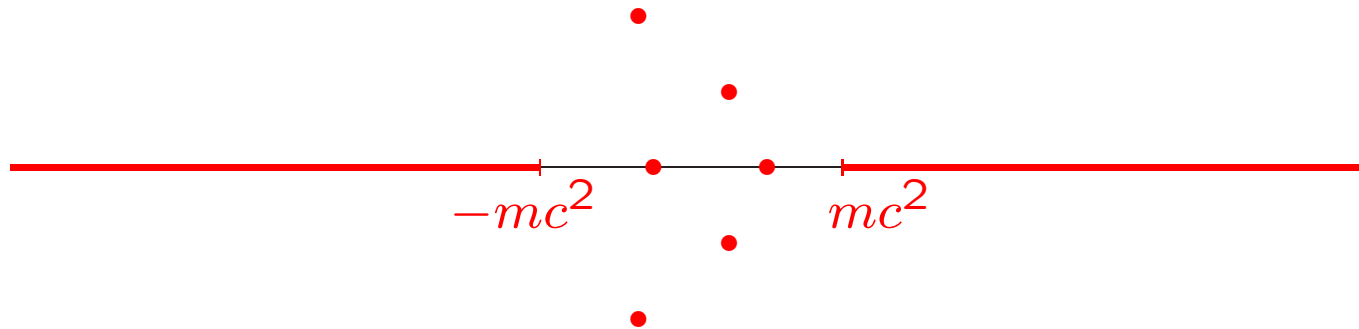
$$\rightsquigarrow \underbrace{\left(-(\lambda - V)^2 + A_0 + \gamma^2 \right)}_{T(\lambda)} u = 0$$

T operator function in $L^2(\mathbb{R}^3)$.

Spectrum of T

$$\sigma_{\text{ess}}(T) = (-\infty, -mc^2] \cup [mc^2, \infty).$$

The non-real spectrum consists only of finitely many complex conjugate pairs of eigenvalues.



Generalised Rayleigh functional

For $u \in H^1(\mathbb{R}^3)$, $u \neq 0$, let $p_{\pm}(u)$ be the zeros of $\lambda \mapsto (T(\lambda)u, u)$ if these are real;

otherwise, set $p_+(u) = -\infty$, $p_-(u) = +\infty$.

$$p_{\pm}(u) = (Vu, u) \pm \sqrt{(Vu, u)^2 - \|Vu\|^2 + \|(A_0 + \gamma^2)^{1/2}u\|^2}$$

$$\text{if } \|u\| = 1 \quad \text{and} \quad p_{\pm}(Cu) = p_{\pm}(u), \quad C \neq 0$$

Theorem. [M.L., Tretter 2006]

Set $\nu := \sup\{p_-(u) : x \in H^1(\mathbb{R}^3), p_-(u) \neq +\infty\}$
and assume that $\nu < mc^2$.

Let $\alpha \in (\nu, \lambda_e)$ and κ be the number of neg. eigenvalues of $T(\alpha)$.

Then $\sigma(T) \cap (\alpha, \lambda_e)$ consists of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$ and

$$\lambda_n = \min_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n + \kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u).$$

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If there are infinitely many eigenvalues, then $\lim_{n \rightarrow \infty} \lambda_n = mc^2$.

If there are $N < \infty$ eigenvalues, then

$$\inf_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n + \kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u) = mc^2 \quad \text{for } n > N$$

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pairs of non-real eigenvalues $\leq \kappa$

In a similar way one can characterise eigenvalues $< \alpha$ with p_- .

Comparison with eigenvalues of a Schrödinger operator

Consider the corresponding Schrödinger operator

$$H = -\frac{1}{2m} \left(-i \hbar \nabla - \frac{e}{c} \vec{A} \right)^2 + V$$

and its negative eigenvalues $\mu_1 \leq \mu_2 \leq \dots$. Then

$$\lambda_{n-\kappa_+}^+ - mc^2 \leq \mu_n.$$

If $\vec{A} = 0$, then

$$\lim_{c \rightarrow \infty} (\lambda_{n-\kappa_+}^+ - mc^2) = \mu_n.$$

Triple variational principle

Let T be defined on $[\alpha, \beta)$ with $\mathcal{D} = \text{dom}(T(\lambda))$ and $\alpha \in \rho(T)$.

Set

$$\mathbf{M}_\alpha^+ := \left\{ \mathcal{M} \subset \mathcal{D} : \mathcal{M} \text{ is a maximal } (T(\alpha) \cdot, \cdot)\text{-non-negative subspace of } \mathcal{D} \right\}.$$

\mathcal{M} is $(T(\alpha) \cdot, \cdot)$ -non-negative if $(T(\alpha)x, x) \geq 0$ for all $x \in \mathcal{M}$.

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Theorem. [M.L., Strauss; cf. also Eschwé, H. Langer 2002]

Assume that the functions $(T(\cdot)x, x)$ are continuous and decreasing at value 0 for all $x \in \mathcal{D}$, $x \neq 0$.

Let $\lambda_1 \leq \lambda_2 \leq \dots$ be eigenvalues of T in $(\alpha, \min \sigma_{\text{ess}}(T))$. Then

$$\lambda_n \geq \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \sup_{\substack{L \subset \mathcal{M} \\ \dim L = n-1}} \inf_{\substack{x \perp L \\ x \in \mathcal{M}, x \neq 0}} p(x).$$

Theorem (continued).

Let T be differentiable and assume in addition that the Virozub–Matsaev (VM) condition is satisfied. Then

$$\min \sigma_{\text{ess}}(T) \geq \sup_{\mathcal{M} \in \mathbf{M}_{\alpha}^{+}} \sup_{\substack{L \subset \mathcal{M} \\ \dim L = n-1}} \inf_{\substack{x \perp L \\ x \in \mathcal{M}, x \neq 0}} p(x), \quad n = 1, 2, \dots$$

In particular,

$$\min \sigma(T) \geq \sup_{\mathcal{M} \in \mathbf{M}_{\alpha}^{+}} \inf_{x \in \mathcal{M}, x \neq 0} p(x).$$

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In particular,

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Corollary. Let $\mu \in (\alpha, \beta)$ and assume that (VM) is satisfied. If there exists a maximal $(T(\alpha) \cdot, \cdot)$ -non-negative subspace \mathcal{M} such that

$$(T(\mu)x, x) \geq 0, \quad \text{for all } x \in \mathcal{M},$$

then $[\alpha, \mu) \subset \rho(T)$.

Theorem. [M.L., Strauss]

If T is an analytic operator function whose values are bounded operators and which satisfies the (VM) condition, then

$$\lambda_n = \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \sup_{\substack{LC\mathcal{M} \\ \dim L = n-1}} \inf_{\substack{x \perp L \\ x \in \mathcal{D}, x \neq 0}} p(x).$$

Idea of proof: choose spectral subspace for the operator function obtained from a linearisation of T in [H. Langer, Markus, Matsaev 2000, 2006].

Block operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

where A and D are self-adjoint operators.

Introduce the Schur complement

$$T(\lambda) = A - \lambda - B(D - \lambda)^{-1}B^*, \quad \lambda \in \rho(D).$$

On $\rho(D)$ the spectra of \mathcal{A} and T coincide, i.e.

$$\sigma(\mathcal{A}) \cap \rho(D) = \sigma(T).$$

Theorem. [M.L., Strauss]

Assume that D is bounded with $d_+ := \max \sigma(D)$,
 A is bounded from below, $\text{dom}(|A|^{1/2}) \subset \text{dom}(B^*)$ and

$$\|B^*x\|^2 \leq a\|x\|^2 + b(Ax, x), \quad x \in \text{dom}(A).$$

for some $a, b \geq 0$.

Let $d_+ < \alpha < \beta$, $(\alpha, \beta) \subset \rho(A)$ and set

$$\hat{\alpha} := \frac{\alpha + d_+}{2} + \sqrt{\left(\frac{\alpha - d_+}{2}\right)^2 + a + b\alpha}$$

If $\hat{\alpha} < \beta$, then $(\hat{\alpha}, \beta) \subset \rho(A)$.

Moreover, equality in variational principle holds for eigenvalues in certain gaps of the essential spectrum.

Theorem. [M.L., Strauss]

\mathcal{A} as above.

Let $d_+ < \alpha < \beta$, and assume that $(\alpha, \beta) \cap \sigma_{\text{ess}}(A) = \emptyset$. Set

$$\hat{\alpha} := \frac{\alpha + d_+}{2} + \sqrt{\left(\frac{\alpha - d_+}{2}\right)^2 + a + b\alpha}$$

If $\hat{\alpha} < \beta$, then $(\hat{\alpha}, \beta) \cap \sigma_{\text{ess}}(\mathcal{A}) = \emptyset$.