Parameterization of Partial Isometries[1]

Motivation: To compute examples for recent research results requires to find column-orthogonal matrices $X \in \mathbb{R}^{N\times n}, N > n$. The problem sounds quite simple:

Write a MATLAB code, say `X=partiso(y,N,n)`, such that for any column-orthogonal matrix $X \in \mathbb{R}^{N\times n}$ there exists a vector of parameters $y \in \mathbb{R}^{Nn-n(n+1)/2}$ with $X = \text{partiso}(y, N, n)$.

It is well-known that $N(N-1)/2$ real parameters are sufficient to generate any orthogonal matrix $\Theta \in \mathbb{R}^{N\times N}$. The number of parameters is equal to the total number of parameters less the number of constraints. Θ has N^2 parameters and there are $N(N+1)/2$ constraints: N from normalizing the N vectors and $N(N-1)/2$ from their mutual orthogonality, hence

$$p_{N,N} := N^2 - N(N+1)/2 = N(N-1)/2.$$

Such an orthogonal matrix can be generated numerically in various ways (for example by Jacobi rotations) but for rectangular orthogonal matrices the situation is not that straightforward.

Recall that a matrix $X \in \mathbb{R}^{N\times n}, N \geq n$, is called a **partial isometry** if $XX^TX = X$ (here the superscript 'T' denotes transposition). In particular we have $X^TX = I_n$ and XX^T is a projector.

The n column vectors of X are normalized and they are mutually perpendicular. Hence there are

$$p_{N,n} = Nn - n(n+1)/2$$

real parameters $y \in \mathbb{R}^{p_{N,n}}$. The question is: given y, how to generate X ?

Special Case - n divides N: Suppose $N = nm$ and consider the partition $X = \begin{bmatrix} X_1 \\ Y \end{bmatrix}$ where $X_1 \in \mathbb{R}^{n\times n}$. Then $X^TX = I_n = X_1^TX_1 + Y^TY$ implies $X_1 = \Phi_1 S_1 \Theta_1, Y = \Psi C_1 \Theta_1$, where $S_1 = \sin(\Sigma_1), C_1 = \cos(\Sigma_1)$ for any diagonal matrix Σ, and $\Phi_1, \Theta_1 \in \mathbb{R}^{n\times n}$ are orthogonal whilst $\Psi \in \mathbb{R}^{n(m-1)\times n}$ is a partial isometry. Let us indicate the number of partitions by writing $X = X_{(m)}$ then the above reads

$$X_{(m)} = \begin{bmatrix} \Phi_1 S_1 \Theta_1 \\ X_{(m-1)} C_1 \Theta_1 \end{bmatrix}$$

which reveals a recursive formula

$$X_{(m-i)} = \begin{bmatrix} \Phi_i S_i \Theta_i \\ X_{(m-1-1)} C_i \Theta_i \end{bmatrix}.$$

The recursion ends if $i = m-1$ and leads to the following expression:

$$X_m = \begin{bmatrix} \Phi_1 S_1 \Theta_1 \\ \Phi_2 S_2 \Theta_2 C_1 \Theta_1 \\ \Phi_3 S_3 \Theta_3 C_2 \Theta_2 C_1 \Theta_1 \\ \vdots \\ \Phi_{m-2} S_{m-2} \Theta_{m-2} C_{m-3} \Theta_{m-3} \cdots C_3 \Theta_3 C_2 \Theta_2 C_1 \Theta_1 \\ \Phi_{m-1} S_{m-1} \Theta_{m-1} C_{m-2} \Theta_{m-2} C_{m-3} \Theta_{m-3} \cdots C_3 \Theta_3 C_2 \Theta_2 C_1 \Theta_1 \\ \Phi_m C_{m-1} \Theta_{m-1} C_{m-2} \Theta_{m-2} C_{m-3} \Theta_{m-3} \cdots C_3 \Theta_3 C_2 \Theta_2 C_1 \Theta_1 \end{bmatrix}.$$

[1] partial-iso.tex

Parameter Count: The above expression consists of $2m-1$ orthogonal matrices $\Psi_i, i = 1, \cdots, m, \Theta_k, k = 1, \cdots, m-1$, which correspond to $(2m-1)n(n-1)/2$ parameters. The pairs $(S_k, C_k), k = 1, \cdots, m-1$, add $(m-1)n$ parameters for a total of

$$p_{nm,n} = (2m-1)n(n-1)/2 + (m-1)n = mn^2 - n(n+1)/2$$

which corresponds to the difference of the n^2m parameters in $X_{(m)}$ and the $n(n+1)/2$ constraints on its column vectors (n form normalization and $n(n-1)/2$ from the mutual orthogonality).

General Case Finally we address the general case $X \in \mathbb{R}^{N\times n}$. It is sufficient to assume that $N = n + r$ with $0 < r < n$ and hence $X = \begin{bmatrix} Y \\ Z^T \end{bmatrix}$ where $Y \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{r\times n}$. Now we have

$$X^TX = Y^TY + ZZ^T = \Theta^T C^2 \Theta + \Theta^T S^2 \Theta = I_n.$$

Because rank $(Y) \leq r$ we have $S = \mathrm{diag}\left(\widehat{S}, 0\right), C = \mathrm{diag}\left(\widehat{C}, E\right)$ with $\widehat{S} = \mathrm{diag}\left(\sin(\sigma_1), \cdots, \sin(\sigma_r)\right)$, $\widehat{C} = \mathrm{diag}\left(\cos(\sigma_1), \cdots, \cos(\sigma_r)\right)$ and a diagonal matrix $E \in \mathbb{R}^{(n-r)\times(n-r)}$ such that $E^2 = I_{n-r}$. Hence the result reads

$$X = \begin{bmatrix} \Psi \mathrm{diag}\left(\widehat{C}, I_{n-r}\right)\Theta \\ \Phi\widehat{S}\widehat{\Theta} \end{bmatrix}$$

with the orthogonal matrix $\Phi \in \mathbb{R}^{r\times r}$, with $\widehat{\Theta} := [I_r \; 0]\Theta$ and E has been absorbed by the orthogonal matrix $\Psi \in \mathbb{R}^{n\times n}$. Note that we have the freedom to insert any orthogonal matrix $\Theta_0 \in \mathbb{R}^{(n-r)\times(n-r)}$ into the first partition:

$$\Psi \mathrm{diag}\left(I_r, \Theta_0^T\right) \mathrm{diag}\left(\widehat{C}, I_{n-r}\right) \mathrm{diag}\left(I_r, \Theta_0\right)\Theta.$$

Parameter Count: The orthogonal matrices Ψ and Θ contribute $n(n-1)$ parameters, the pair $(\widehat{C}, \widehat{S})$ add r parameters, $r(r-1)/2$ parameters are from the the orthogonal matrix Ψ and the freedom to choose any Θ_0 reduces the a total by $(n-r)(n-r-1)/2$ which yields

$$p_{n+r,n} = n(n-1) + r + r(r-1)/2 - (n-r)(n-r-1)/2 = n(n-1)/2 + rn = (n+r)n - n(n+1)/2.$$

This result corresponds to the difference of $(n+r)n$ parameters of $X \in \mathbb{R}^{(n+r)\times n}$ and the $n(n+1)/2$ constraints of its column vectors.

Conclusion: Although the parameter count is correct, the above expression is unsuitable for coding because of the 'redundant' matrix Θ_0. Hence I consider this as an 'open problem'. The following papers have been consulted but did not help to solve the problem:

- X. Sun & Chr. Bischof (1995), A Basis-Kernel Representation of Orthogonal Matrices, SIAM Journal on Matrix Analysis and Applications, Volume 16 , Issue 4, Pages: 1184 - 1196.

- C. C. Paige & M. Wei, History and Generality of the CS Decomposition (1994), Linear Algebra and its Applications, Volume 208-209, Pages: 303 - 326.