A *Homogeneous Coordinates* View of Eigenvalues in the Quadratic Eigenvalue Problem

S. D. Garvey

Contents

- Matrix Polynomials and their *L.A.M.s.*
- SPTs and Diagonalising SPTs
- The L.A.M.s of (1×1) systems.
- Homogeneous Coordinates.
- The Eigenvectors in context of H.C.
- An Algorithm.

Matrices	:	Emboldened capital Roman … **A**, **B**, **C**, etc.
Vectors	:	Emboldened lowercase Roman … **a**, **b**, etc.
Scalars	:	Italicised lowercase. Mostly Greek … α, β, etc.
		(the exceptions are s (Laplace frequency) and t (time))

Matrix Polynomials and their LAMs.

Main focus: *Second Order Systems*

For context, also look at *first*- and *third- order*.

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M} + s^3\mathbf{N}\right)\mathbf{q} = \mathbf{f}$$ Third Order

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M}\right)\mathbf{q} = \mathbf{f}$$ Second Order

$$\left(\mathbf{K} + s\mathbf{D}\right)\mathbf{q} = \mathbf{f}$$ First Order

All system matrices have dimension ($a \times a$)

Matrix Polynomials and their LAMs.

The general ℓ^{th}-*order* matrix polynomial has $(\ell+1)$ *Lancaster Augmented Matrices*. $(\ell a \times \ell a)$.

$$\underline{\mathbf{N}} = \left[\begin{array}{ccc|} \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{K} & \mathbf{D} \\ \mathbf{K} & \mathbf{D} & \mathbf{M} \\ \hline \end{array}\right] \qquad \underline{\mathbf{M}} = \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{K} & \mathbf{0} \\ \mathbf{K} & \mathbf{D} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\mathbf{N} \end{array}\right]$$

3rd Order

$$\underline{\mathbf{D}} = \left[\begin{array}{c|cc} \mathbf{K} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{M} & -\mathbf{N} \\ \mathbf{0} & -\mathbf{N} & \mathbf{0} \end{array}\right] \qquad \underline{\mathbf{K}} = -\left[\begin{array}{|ccc} \hline \mathbf{D} & \mathbf{M} & \mathbf{N} \\ \mathbf{M} & \mathbf{N} & \mathbf{0} \\ \mathbf{N} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

Naming convention:- which system matrix is *omitted*.

Matrix Polynomials and their LAMs.

2nd Order

$$\underline{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \qquad \underline{\mathbf{D}} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \qquad \underline{\mathbf{K}} = -\begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}$$

1st Order

$$\underline{\mathbf{D}} = \begin{bmatrix} \mathbf{K} \end{bmatrix} \qquad \underline{\mathbf{K}} = \begin{bmatrix} -\mathbf{D} \end{bmatrix}$$

Naming convention:- which *system* matrix is *omitted*.

SPTs and *Diagonalising* SPTs.

For present purposes, a *Structure Preserving Transformation* is a transformation which maps ALL OF the LAMs of some original system to the corresponding LAMs of some new system

$$\underline{\mathbf{T}_L}^T \, \underline{\mathbf{X}_0} \, \underline{\mathbf{T}_R} = \underline{\mathbf{X}_1}$$

SPTs and *Diagonalising* SPTs.

3rd Order

$$\underline{\mathbf{T}_L}^T \left[\begin{array}{ccc|} \mathbf{0} & \mathbf{0} & \mathbf{K}_0 \\ \mathbf{0} & \mathbf{K}_0 & \mathbf{D}_0 \\ \mathbf{K}_0 & \mathbf{D}_0 & \mathbf{M}_0 \\ \hline \end{array}\right] \underline{\mathbf{T}_R} = \left[\begin{array}{ccc|} \mathbf{0} & \mathbf{0} & \mathbf{K}_1 \\ \mathbf{0} & \mathbf{K}_1 & \mathbf{D}_1 \\ \mathbf{K}_1 & \mathbf{D}_1 & \mathbf{M}_1 \\ \hline \end{array}\right] \qquad \underline{\mathbf{T}_L}^T \underline{\mathbf{N}_0}\, \underline{\mathbf{T}_R} = \underline{\mathbf{N}_1}$$

$$\underline{\mathbf{T}_L}^T \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{K}_0 & \mathbf{0} \\ \mathbf{K}_0 & \mathbf{K}_0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\mathbf{N}_0 \end{array}\right] \underline{\mathbf{T}_R} = \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{K}_1 & \mathbf{K}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\mathbf{N}_1 \end{array}\right] \qquad \underline{\mathbf{T}_L}^T \underline{\mathbf{M}_0}\, \underline{\mathbf{T}_R} = \underline{\mathbf{M}_1}$$

$$\underline{\mathbf{T}_L}^T \left[\begin{array}{c|cc} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{M}_0 & -\mathbf{N}_0 \\ \mathbf{0} & -\mathbf{N}_0 & \mathbf{0} \end{array}\right] \underline{\mathbf{T}_R} = \left[\begin{array}{c|cc} \mathbf{K}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{M}_1 & -\mathbf{N}_1 \\ \mathbf{0} & -\mathbf{N}_1 & \mathbf{0} \end{array}\right] \qquad \underline{\mathbf{T}_L}^T \underline{\mathbf{D}_0}\, \underline{\mathbf{T}_R} = \underline{\mathbf{D}_1}$$

$$\underline{\mathbf{T}_L}^T \left[\begin{array}{|ccc} \hline \mathbf{D}_0 & \mathbf{M}_0 & \mathbf{N}_0 \\ \mathbf{M}_0 & \mathbf{N}_0 & \mathbf{0} \\ \mathbf{N}_0 & \mathbf{0} & \mathbf{0} \end{array}\right] \underline{\mathbf{T}_R} = \left[\begin{array}{|ccc} \hline \mathbf{D}_1 & \mathbf{M}_1 & \mathbf{N}_1 \\ \mathbf{M}_1 & \mathbf{N}_1 & \mathbf{0} \\ \mathbf{N}_1 & \mathbf{0} & \mathbf{0} \end{array}\right] \qquad \underline{\mathbf{T}_L}^T \underline{\mathbf{K}_0}\, \underline{\mathbf{T}_R} = \underline{\mathbf{K}_1}$$

SPTs and *Diagonalising* SPTs.

2nd Order

$$\underline{\mathbf{T}}_L^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_0 \\ \mathbf{K}_0 & \mathbf{K}_0 \end{bmatrix} \underline{\mathbf{T}}_R = \begin{bmatrix} \mathbf{0} & \mathbf{K}_1 \\ \mathbf{K}_1 & \mathbf{K}_1 \end{bmatrix} \qquad \underline{\mathbf{T}}_L^T \underline{\mathbf{M}}_0 \underline{\mathbf{T}}_R = \underline{\mathbf{M}}_1$$

$$\underline{\mathbf{T}}_L^T \left[\begin{array}{c|c} \mathbf{K}_0 & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{M}_0 \end{array}\right] \underline{\mathbf{T}}_R = \left[\begin{array}{c|c} \mathbf{K}_1 & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{M}_1 \end{array}\right] \qquad \underline{\mathbf{T}}_L^T \underline{\mathbf{D}}_0 \underline{\mathbf{T}}_R = \underline{\mathbf{D}}_1$$

$$\underline{\mathbf{T}}_L^T \begin{bmatrix} -\mathbf{D}_0 & -\mathbf{M}_0 \\ -\mathbf{M}_0 & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}}_R = \begin{bmatrix} -\mathbf{D}_1 & -\mathbf{M}_1 \\ -\mathbf{M}_1 & \mathbf{0} \end{bmatrix} \qquad \underline{\mathbf{T}}_L^T \underline{\mathbf{K}}_0 \underline{\mathbf{T}}_R = \underline{\mathbf{K}}_1$$

IF $\underline{\mathbf{T}}_L$ and $\underline{\mathbf{T}}_R$ both invertible, then this SPT is a *Structure Preserving Equivalence* (SPE).

SPTs and *Diagonalising* SPTs.

1st Order

$$\mathbf{T}_L^T \left[\mathbf{K}_0 \right] \mathbf{T}_R = \left[\mathbf{K}_1 \right] \qquad \underline{\mathbf{T}_L}^T \underline{\mathbf{D}_0}\, \underline{\mathbf{T}_R} = \underline{\mathbf{D}_1}$$

$$\mathbf{T}_L^T \left[-\mathbf{D}_0 \right] \mathbf{T}_R = \left[-\mathbf{D}_1 \right] \qquad \underline{\mathbf{T}_L}^T \underline{\mathbf{K}_0}\, \underline{\mathbf{T}_R} = \underline{\mathbf{K}_1}$$

Even 1st order systems have SPTs – though they are trivial.

SPTs and *Diagonalising* SPTs.

A given SPT is *diagonalising* if ALL of the system matrices of the new system are *diagonal*.

$$\{\mathbf{K}_0, \mathbf{D}_0, \mathbf{M}_0, \mathbf{N}_0\} \xRightarrow{\{\mathbf{T}_L, \mathbf{T}_R\}} \{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1, \mathbf{N}_1\}$$

with $\left(\mathbf{K}_1 + s\mathbf{D}_1 + s^2\mathbf{M}_1 + s^3\mathbf{N}_1\right)$ *diagonal* for all s.

SPTs and *Diagonalising* SPTs.

Most conventional approaches to solving for the eigenvalues of a matrix polynomial begin with a *Linearisation* to transform the problem into a *Generalised Eigenvalue Problem*. The LAMs provide a particularly attractive vector space of linearisations[¥].

$$\left(\mathbf{K} + \lambda\mathbf{D} + \lambda^2\mathbf{M} + \lambda^3\mathbf{N}\right)v = 0$$

3rd Order Matrix Poly.

$$\left(\begin{bmatrix} \mathbf{K} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} & -\mathbf{N} \\ \mathbf{0} & -\mathbf{N} & \mathbf{0} \end{bmatrix} - \lambda \begin{bmatrix} -\mathbf{D} & -\mathbf{M} & -\mathbf{N} \\ -\mathbf{M} & -\mathbf{N} & \mathbf{0} \\ -\mathbf{N} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right)\begin{bmatrix} \mathbf{v} \\ \mathbf{v}\lambda \\ \mathbf{v}\lambda^2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

¥ Mackey, D. S.; Mackey, N.; Mehl, C; Mehrmann, V. 'Vector Spaces of Linearizations for Matrix Polynomials' *SIAM J. of Matrix Analysis & Applns*, 2006, 28(4), pp971-1004

SPTs and *Diagonalising* SPTs.

Some choices of linearisation are good for particular problems and some not so good[¥].

If leading and trailing coefficients are both invertible, then all LAMs are non-singular. Then

$$\det\left(\mathbf{K} + \lambda\mathbf{D} + \lambda^2\mathbf{M} + \lambda^3\mathbf{N}\right) = 0$$

3rd Order

… is equivalent to any one of …

$$\det\left(\underline{\mathbf{N}} - \lambda\underline{\mathbf{M}}\right) = 0 \qquad \det\left(\underline{\mathbf{M}} - \lambda\underline{\mathbf{D}}\right) = 0 \qquad \det\left(\underline{\mathbf{D}} - \lambda\underline{\mathbf{K}}\right) = 0$$

¥ Higham, N.J.; Mackey, D.S.; Tisseur, F. & Garvey, S. D. ‘Scaling, Sensitivity and Stability in the Numerical Solution of Quadratic Eigenvalue Problems’. *IJNME,* 73(3). pp344 – 360

SPTs and *Diagonalising* SPTs.

$$\det\left(\mathbf{K} + \lambda\mathbf{D} + \lambda^2\mathbf{M}\right) = 0$$

2nd Order

… is equivalent to either of …

$$\det\left(\underline{\mathbf{M}} - \lambda\underline{\mathbf{D}}\right) = 0 \qquad \det\left(\underline{\mathbf{D}} - \lambda\underline{\mathbf{K}}\right) = 0$$

$$\det\left(\mathbf{K} + \lambda\mathbf{D}\right) = 0$$

1st Order

… is equivalent to … $\det\left(\underline{\mathbf{D}} - \lambda\underline{\mathbf{K}}\right) = 0$

SPTs and *Diagonalising* SPTs.

ALL polynomials whose Jordan form is diagonal can be *diagonalised* by SPTs (proof – by construction).

ALL 2nd-order polynomials whose Jordan form contains no Jordan block of dimension>2 are diagonalisable by SPT [¥]

ALL 3rd-order polynomials whose Jordan form contains no Jordan block of dimension>3 are diagonalisable by SPT

ALL $\mathbf{l}^{th}$-order polynomials whose Jordan form contains no Jordan block of dimension>$\mathbf{l}$ are diagonalisable by SPT

¥ Lancaster, P. & Zaballa, I. 'Diagonalisable Quadratic Eigenvalue Problems'. *Mechanical Systems and Signal Processing,* 2009, 23(4). p1134 – 1144

SPTs and *Diagonalising* SPTs.

Two short *asides*.

(1) Diagonalising SPTs are not unique.

(2) At present, finding a diagonalising SPT (nearly[¥]) always begins with solving a generalised EVP

$$\underline{\mathbf{U}_L}^T\left(\underline{\mathbf{A}} - \lambda\underline{\mathbf{B}}\right)\underline{\mathbf{U}_R} = \left(\ddot{\mathbf{E}} - \lambda\mathbf{I}\right)$$

Matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ can be chosen as <u>any</u> independent linear combinations of the LAMs. Finding $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$ involves **l** decoupled problems.

The *anti-eigenvalue problem*! Find $\{e, f, g, h\}$ such that $\underline{\mathbf{B}} = \left(e\underline{\mathbf{N}} + f\,\underline{\mathbf{M}} + g\,\underline{\mathbf{D}} + h\underline{\mathbf{K}}\right)$ well-conditioned.

¥ Chu, M.T. & Del Buono, N. 'Total Decoupling for a General Quadratic Pencil. Part II: Structure Preserving Isospectral Flows'. *Journal of Sound & Vibration*, 2008, 309(1-2), pp112-118

The LAMs of (1 × 1) systems

$$\left(k + sd + s^2 m + s^3 n\right) q = f$$

Third Order

The LAMs of this system are …

$$\underline{\mathbf{N}} = \begin{bmatrix} 0 & 0 & k \\ 0 & k & d \\ k & d & m \end{bmatrix} \qquad \underline{\mathbf{M}} = \left[\begin{array}{cc|c} 0 & k & 0 \\ k & d & 0 \\ \hline 0 & 0 & -n \end{array}\right]$$

$$\underline{\mathbf{D}} = \left[\begin{array}{c|cc} k & 0 & 0 \\ \hline 0 & -m & -n \\ 0 & -n & 0 \end{array}\right] \qquad \underline{\mathbf{K}} = -\begin{bmatrix} d & m & n \\ m & n & 0 \\ n & 0 & 0 \end{bmatrix}$$

The LAMs of (1 × 1) systems

Observe that … **Third Order**

$$n\begin{bmatrix}0 & 0 & k\\ 0 & k & d\\ k & d & m\end{bmatrix}+m\begin{bmatrix}0 & k & 0\\ k & d & 0\\ 0 & 0 & -n\end{bmatrix}+d\begin{bmatrix}k & 0 & 0\\ 0 & -m & -n\\ 0 & -n & 0\end{bmatrix}+k\begin{bmatrix}-d & -m & -n\\ -m & -n & 0\\ -n & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

More concisely: $\left(n\underline{\mathbf{N}}+m\underline{\mathbf{M}}+d\underline{\mathbf{D}}+k\underline{\mathbf{K}}\right)=\mathbf{0}$

The LAMs of (1 × 1) systems

$$\left(k + sd + s^2 m\right) q = f$$

Second Order

The LAMs of this system are …

$$\underline{\mathbf{M}} = \begin{bmatrix} 0 & k \\ k & d \end{bmatrix} \qquad \underline{\mathbf{D}} = \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix} \qquad \underline{\mathbf{K}} = \begin{bmatrix} -d & -m \\ -m & 0 \end{bmatrix}$$

Observe that

$$m\begin{bmatrix} 0 & k \\ k & d \end{bmatrix} + d\begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix} + k\begin{bmatrix} -d & -m \\ -m & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

... or more compactly …

$$m\underline{\mathbf{M}} + d\underline{\mathbf{D}} + k\underline{\mathbf{K}} = \mathbf{0}$$

The LAMs of (1 × 1) systems

$$\left(k + sd\right)q = f$$

First Order

The LAMs of this system are …

$$\underline{\mathbf{D}} = \left[k\right] \qquad \underline{\mathbf{K}} = \left[-d\right]$$

Trivially … $d\left[k\right] + k\left[-d\right] = \left[0\right]$

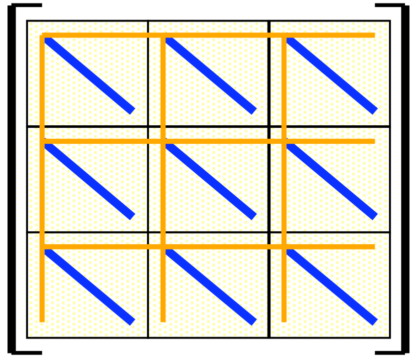
... or more compactly … $d\,\underline{\mathbf{D}} + k\,\underline{\mathbf{K}} = \mathbf{0}$

Homogeneous Coordinates.

Now suppose that a diagonalising SPT exists for some $(a \times a)$ third order system:

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M} + s^3\mathbf{N}\right)\mathbf{q} = \mathbf{f} \qquad \text{Third Order}$$

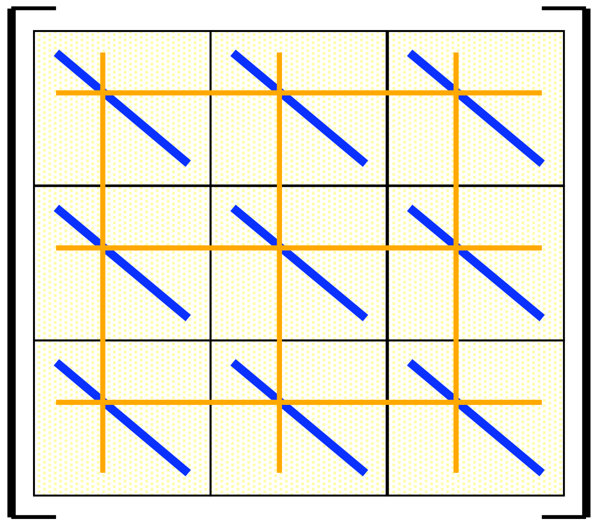
$$\left.\begin{aligned}
\underline{\mathbf{T}_L}^T \underline{\mathbf{K}_0}\, \underline{\mathbf{T}_R} &= \underline{\mathbf{K}_1} \\
\underline{\mathbf{T}_L}^T \underline{\mathbf{D}_0}\, \underline{\mathbf{T}_R} &= \underline{\mathbf{D}_1} \\
\underline{\mathbf{T}_L}^T \underline{\mathbf{M}_0}\, \underline{\mathbf{T}_R} &= \underline{\mathbf{M}_1} \\
\underline{\mathbf{T}_L}^T \underline{\mathbf{N}_0}\, \underline{\mathbf{T}_R} &= \underline{\mathbf{N}_1}
\end{aligned}\right\}$$



All of the *diagonalised* LAMs have above "diagonal-blocks" structure. The *diagonalised* system comprises *a* decoupled (1×1) systems.

Homogeneous Coordinates

Third Order case ../cntd.



Let $\{k_i, d_i, m_i, n_i\}$ represent the i^{th} diagonal entries of diagonal 3rd-order system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1, \mathbf{N}_1\}$ respectively.

It is clear that

$$\dim\left(\ker\left(n_i\underline{\mathbf{N}_1} + m_i\underline{\mathbf{M}_1} + d_i\underline{\mathbf{D}_1} + k_i\underline{\mathbf{K}_1}\right)\right) \geq 3$$

IF $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$ invertible, it follows instantly that

$$\dim\left(\ker\left(n_i\underline{\mathbf{N}_0} + m_i\underline{\mathbf{M}_0} + d_i\underline{\mathbf{D}_0} + k_i\underline{\mathbf{K}_0}\right)\right) \geq 3$$

Homogeneous Coordinates.

Theorem about the eigenvalues of 3rd Order systems

If {**K**, **D**, **M**, **N**} are the matrices of a 3rd-order system which is *diagonalisable* by SPE, then these two statements are equivalent

$$\det\left(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^{\,2} \mathbf{M} + \lambda_i^{\,3} \mathbf{N}\right) = 0 \qquad for \quad i = 1,2,3$$

$$k + \lambda_i d + \lambda_i^{\,2} m + \lambda_i^{\,3} n = 0 \qquad for \quad i = 1,2,3$$

where

$$\dim\left(\ker\left(n\underline{\mathbf{N}} + m\underline{\mathbf{M}} + d\,\underline{\mathbf{D}} + k\underline{\mathbf{K}}\right)\right) = 3$$

The "IF" condition is probably unnecessary – it facilitates proof.

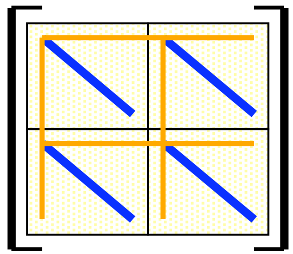
Homogeneous Coordinates.

Now suppose that a diagonalising SPT exists for some $(a \times a)$ second order system:

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M}\right)\mathbf{q} = \mathbf{f}$$

Second Order

$$\left.\begin{aligned}
\underline{\mathbf{T}_L}^T\,\underline{\mathbf{K}_0}\,\underline{\mathbf{T}_R} &= \underline{\mathbf{K}_1} \\
\underline{\mathbf{T}_L}^T\,\underline{\mathbf{D}_0}\,\underline{\mathbf{T}_R} &= \underline{\mathbf{D}_1} \\
\underline{\mathbf{T}_L}^T\,\underline{\mathbf{M}_0}\,\underline{\mathbf{T}_R} &= \underline{\mathbf{M}_1}
\end{aligned}\right\}$$

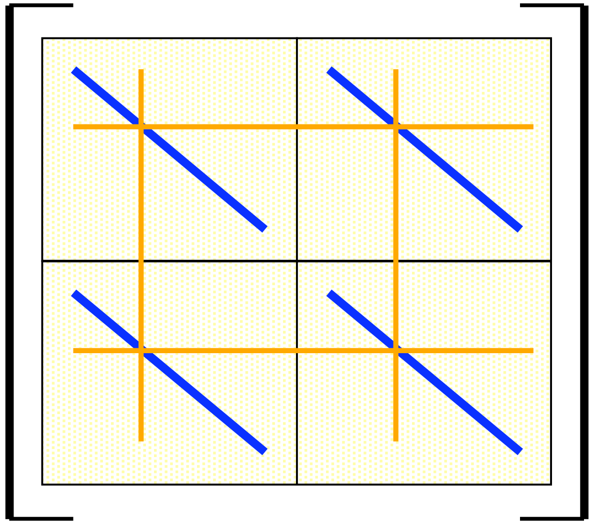


All three *diagonalised* LAMs have above “diagonal-blocks” structure.

The *diagonalised* system comprises a decoupled (1×1) systems.

Homogeneous Coordinates

Second-Order case ../cntd.



Let $\{k_i, d_i, m_i\}$ represent the i^{th} diagonal entries of diagonal 2nd-order system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ respectively.

It is clear that

$$\dim\left(\ker\left(m_i\underline{\mathbf{M}_1} + d_i\underline{\mathbf{D}_1} + k_i\underline{\mathbf{K}_1}\right)\right) \geq 2$$

IF $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$ invertible, it follows instantly that

$$\dim\left(\ker\left(m_i\underline{\mathbf{M}_0} + d_i\underline{\mathbf{D}_0} + k_i\underline{\mathbf{K}_0}\right)\right) \geq 2$$

¥ Abuazoum, L. & Garvey, S.D. 'Eigenvalue and Eigenvector Derivatives using Structure Preserving Equivalences'. To Appear *J. of Sound & Vibration*,

Homogeneous Coordinates.

Theorem about the eigenvalues of 2nd Order systems

If $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are the matrices of a 2nd-order system which is *diagonalisable* by SPE, then these two statements are equivalent

$$\det\left(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i{}^2 \mathbf{M}\right) = 0 \qquad \textit{for} \quad i = 1,2$$

$$k + \lambda_i d + \lambda_i{}^2 m = 0 \qquad \textit{for} \quad i = 1,2$$

where

$$\dim\left(\ker\left(m\underline{\mathbf{M}} + d\,\underline{\mathbf{D}} + k\,\underline{\mathbf{K}}\right)\right) = 2$$

The "IF" condition is probably unnecessary – it facilitates proof.

Homogeneous Coordinates.

The extension to 1st Order systems is obvious

These two statements are equivalent

$$\det(\mathbf{K} + \lambda \mathbf{D}) = 0$$

$$k + \lambda d = 0$$

where

$$\dim(\ker(d\,\underline{\mathbf{D}} + k\,\underline{\mathbf{K}})) = 1$$

Homogeneous Coordinates.

Scaling of the homogeneous coordinates:

Without any loss of generality, we can assert for **3rd-order systems** that the homogeneous coordinates $\{k, d, m, n\}$ representing a *triple* of eigenvalues for the system will be scaled according to:

$$\left(\frac{k}{k_{ref}}\right)^2 + \left(\frac{d}{d_{ref}}\right)^2 + \left(\frac{m}{m_{ref}}\right)^2 + \left(\frac{n}{n_{ref}}\right)^2 = 1$$

A hyper-sphere providing *double-cover* for the eigenvalues.

Homogeneous Coordinates.

Scaling of the homogeneous coordinates:

Without any loss of generality, we can assert for **2nd-order systems** that the homogeneous coordinates $\{k, d, m\}$ representing a *pair* of eigenvalues for the system will be scaled according to:

$$\left(\frac{k}{k_{ref}}\right)^2 + \left(\frac{d}{d_{ref}}\right)^2 + \left(\frac{m}{m_{ref}}\right)^2 = 1$$

A sphere providing *double-cover* for the eigenvalues.

A pair of real angles <u>could</u> be used to represent the pair of eigenvalues (but better to retain three coordinates with one constraint?)

Homogeneous Coordinates.

Scaling of the homogeneous coordinates:

Without any loss of generality, we can assert for **1st-order systems** that the homogeneous coordinates $\{k, d\}$ representing a *single* eigenvalue for the system will be scaled according to:

$$\left(\frac{k}{k_{ref}}\right)^2 + \left(\frac{d}{d_{ref}}\right)^2 = 1$$

A circle providing *double-cover* for the eigenvalues.

A single real angle is used to represent the eigenvalue.

Eigenvectors with the Homog. Coords.

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M}\right)\mathbf{q} = \mathbf{f}$$

Second Order

Given that $\{k, d, m\}$ characterising a pair of eigenvalues the corresponding *eigenvectors* are represented by

$$\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2} d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2} d\mathbf{v}_R\right) \end{bmatrix} \begin{bmatrix} 0 & k \\ k & d \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2} d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2} d\mathbf{v}_R\right) \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \boxed{\begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2} d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2} d\mathbf{v}_R\right) \end{bmatrix}} \begin{bmatrix} -d & -m \\ -m & 0 \end{bmatrix}^{-1}$$

Two REAL vectors here $\{\mathbf{u}_R, \mathbf{v}_R\}$ of length a.
(Matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are all $(a \times a)$).

Eigenvectors with the Homog. Coords.

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M}\right)\mathbf{q} = \mathbf{f}$$

Second Order ../cntd

The *eigenvectors* are equally-well represented by

$$\left(m\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} + d\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + k\begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \right) \boxed{\begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2} d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2} d\mathbf{v}_R\right) \end{bmatrix}}$$

$$= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

The pair of columns above spans the same space of a pair of eigenvectors of a lin<u>stn</u> corresponding to … $\{k, d, m\}$.

Algorithms based on pairs of vectors $\{\mathbf{u}_R, \mathbf{v}_R\}$ (or sets of these pairs) would seem to be possible.

Eigenvectors with the Homog. Coords.

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M}\right)\mathbf{q} = \mathbf{f}$$

Second Order ../cntd

Another view of the (SPE) *eigenvectors* …

$$\begin{bmatrix} \left(\mathbf{u}_L - \tfrac{1}{2}d\mathbf{v}_L\right) & \left(-m\mathbf{v}_L\right) \\ \left(k\mathbf{v}_L\right) & \left(\mathbf{u}_L + \tfrac{1}{2}d\mathbf{v}_L\right) \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2}d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2}d\mathbf{v}_R\right) \end{bmatrix} = \begin{bmatrix} 0 & k \\ k & d \end{bmatrix}$$

$$\begin{bmatrix} \left(\mathbf{u}_L - \tfrac{1}{2}d\mathbf{v}_L\right) & \left(-m\mathbf{v}_L\right) \\ \left(k\mathbf{v}_L\right) & \left(\mathbf{u}_L + \tfrac{1}{2}d\mathbf{v}_L\right) \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2}d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2}d\mathbf{v}_R\right) \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}$$

$$\begin{bmatrix} \left(\mathbf{u}_L - \tfrac{1}{2}d\mathbf{v}_L\right) & \left(-m\mathbf{v}_L\right) \\ \left(k\mathbf{v}_L\right) & \left(\mathbf{u}_L + \tfrac{1}{2}d\mathbf{v}_L\right) \end{bmatrix}^T \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2}d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2}d\mathbf{v}_R\right) \end{bmatrix} = \begin{bmatrix} d & m \\ m & 0 \end{bmatrix}$$

Eigenvectors with the Homog. Coords.

$$\left(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M}\right)\mathbf{q} = \mathbf{f}$$

Second Order ../cntd

Vector $\mathbf{u}_R$ looks like a "privileged member" of the space $[\mathbf{u}_R, \mathbf{v}_R]$. Is it?

No. The concept of an *automorphic* SPT applies.
Choose scalars $\{f, g\}$ such that $(f^2 - g^2(d^2/4 - km)) = 1$.

$$\begin{bmatrix} \left(f - \tfrac{1}{2}gd\right) & \left(-gm\right) \\ \left(gk\right) & \left(f + \tfrac{1}{2}gd\right) \end{bmatrix}^T \begin{bmatrix} 0 & k \\ k & d \end{bmatrix} \begin{bmatrix} \left(f - \tfrac{1}{2}gd\right) & \left(-gm\right) \\ \left(gk\right) & \left(f + \tfrac{1}{2}gd\right) \end{bmatrix} = \begin{bmatrix} 0 & k \\ k & d \end{bmatrix}$$

$$\begin{bmatrix} \left(f - \tfrac{1}{2}gd\right) & \left(-gm\right) \\ \left(gk\right) & \left(f + \tfrac{1}{2}gd\right) \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix} \begin{bmatrix} \left(f - \tfrac{1}{2}gd\right) & \left(-gm\right) \\ \left(gk\right) & \left(f + \tfrac{1}{2}gd\right) \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}$$

etc.

Eigenvectors with the Homog. Coords.

Vector $\mathbf{u}_R$ looks like a "privileged member" of the space $[\mathbf{u}_R, \mathbf{v}_R]$. Is it? …/cntd. (*NO*)

$$\begin{bmatrix} \left(\mathbf{u}_R - \tfrac{1}{2} d\mathbf{v}_R\right) & \left(-m\mathbf{v}_R\right) \\ \left(k\mathbf{v}_R\right) & \left(\mathbf{u}_R + \tfrac{1}{2} d\mathbf{v}_R\right) \end{bmatrix} \begin{bmatrix} \left(f - \tfrac{1}{2} gd\right) & \left(-gm\right) \\ \left(gk\right) & \left(f + \tfrac{1}{2} gd\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\mathbf{u}'_R - \tfrac{1}{2} d\mathbf{v}'_R\right) & \left(-m\mathbf{v}'_R\right) \\ \left(k\mathbf{v}'_R\right) & \left(\mathbf{u}'_R + \tfrac{1}{2} d\mathbf{v}'_R\right) \end{bmatrix}$$

Vectors $\{\mathbf{u}'_R, \mathbf{v}'_R\}$ are distinct from $\{\mathbf{u}_R, \mathbf{v}_R\}$.

Vector $\mathbf{u}'_R$ can be <u>any</u> element of the subspace $[\mathbf{u}_R, \mathbf{v}_R]$.

An Algorithm.

Preliminary remark: if we can find a pair of *eigenvalues* (represented by $\{k, d, m\}$) and a corresponding pair of *eigenvectors* (represented by $[\mathbf{u}_R, \mathbf{v}_R]$, then we have¥ a method by which the QEP can be *deflated*.

¥ Tisseur, F.; Garvey, S.D. & Munro, C. 'Deflating Quadratic Matrix Polynomials with Structure Preserving Transformns' Submitted to LAA, (special issue in honour of P. Stewart)

An Algorithm.

(1) Choose an arbitrary triple $\{k, d, m\}$ defining a point on the sphere. We will find a pair of eigenvalues at a "nearby" point.

(2) Form $\underline{\mathbf{X}} := \left(m\underline{\mathbf{M}} + d\underline{\mathbf{D}} + k\underline{\mathbf{K}}\right)$ and use $\mathbf{w}_R = \underline{\mathbf{X}}^{-1}\mathbf{w}_R$ several times (with orthonormalisation). Also $\mathbf{w}_L = \underline{\mathbf{X}}^{-T}\mathbf{w}_L$ several times. $\mathbf{w}_L$ and $\mathbf{w}_R$ are $(2a \text{ x } 2)$

(3) Split $\mathbf{w}_L$ into 4 a-vectors and use SVD to find 2 orthognormal vectors, $\{\mathbf{u}_L, \mathbf{v}_L\}$ which "nearly" span the space. Same for $\mathbf{w}_R$.

(4) Create a (2 x 2) system, $\{\mathbf{K}_{2\times2}, \mathbf{D}_{2\times2}, \mathbf{M}_{2\times2}\}$, by *projection*.

$$\mathbf{K}_{2\times2} = \begin{bmatrix}\mathbf{u}_L & \mathbf{v}_L\end{bmatrix}^T \mathbf{K} \begin{bmatrix}\mathbf{u}_R & \mathbf{v}_R\end{bmatrix}$$

(5) Find its closest $\{k, d, m\}$ to the original triple. GOTO 2.

CONCLUSION.

A *homogeneous coordinates* definition is proposed for the eigenvalues of quadratic eigenvalue problems (in pairs) & for eigenvalues of cubic eigenvalue problems (in triples).

This definition uses the complete set of ($\mathbf{l}$+1) LAMs for a matrix polynomial of order $\mathbf{l}$ and it is based on what linear combinations of these LAMs have a kernel of dimension $\mathbf{l}$.

A crude algorithm is outlined for finding a single “set” of eigenvalues.

APPENDIX: Demonstration of the Algorithm in MATLAB (3-Point Font!)

```
% --- demo_hc ---
% This script illustrates the use of homogeneous coordinates
% for discovering a pair of eigenvalues from the QEP.
% -----------------------------------------------------------------------

% --- #1  First define the system matrices, {K0, D0, M0}.
K0 = [   125   20   -10    15  ;
          30  285     5   -11  ;
         -12    0   300    50  ;
         -14    4    20   200  ];

D0 = [    10  -20     0     1  ;
         -20    5     2     0  ;
           0   -2     6   -10  ;
           1    0    10     8  ];

M0 = [    30    0    -5     0  ;
           0   10     2     0  ;
          -5    2    20   -10  ;
           0    0   -10    50  ];

ZN = zeros(size(K0)); IN = eye(size(K0));
N  = size(K0,1);
K0, pause;         D0, pause;          M0, pause;

% --- Set up the LAMs for this matrix polynomial.
M__0 = [ ZN  K0; K0  D0];
D__0 = [ K0  ZN; ZN -M0];
K__0 =-[ D0  M0; M0  ZN];


% --- Now prepare to run an iterative search for a pair of eigenvalues
%     (together with the left and right eigenvectors).
irng = (1:N);      jrng = irng+N;      krng = [(2*N-1); 2*N];
Z2   = zeros(2,2); I2   = eye(2,2);

disp(' '); disp(' ');
disp(' * * * Algorithm for finding a pair of QEP roots * * * ');
disp(' '); disp(' ');

disp(' Enter a triple of numbers in the order {k, d, m} to indicate target ');
disp(' pair of eigs. Note: these need not be normalised eg. { 995  110  20 }');
disp('  ');
ktt = input(' Enter target value for <k> : ');
dtt = input(' Enter target value for <d> : ');
mtt = input(' Enter target value for <m> : ');

tx = sqrt(ktt^2 + dtt^2 + mtt^2);      ktt = ktt/tx;
dtt = dtt/tx;                          mtt = mtt/tx;
```

```
% ---
igo = 1;  istep=0;
SVress_reco = zeros(0,6);   % An array of "residual" SVs.
while (igo==1)

    % --- Now form the <target> X and SVD it.
    X = mtt*M__0 + dtt*D__0 + ktt*K__0;
    [U0,S0,V0] = svd(X);     SingVals0 = diag(S0);

    % --- Extract the vectors corresponding to smallest two singular values.
    uv__L = U0(:, krng );    uv__R = V0(:, krng );

    % --- Find two real vectors dominating the four in UV__.
    uv4L  = [ uv__L(irng,1)  uv__L(irng,2)  uv__L(jrng,1)  uv__L(jrng,2) ];
    uv4R  = [ uv__R(irng,1)  uv__R(irng,2)  uv__R(jrng,1)  uv__R(jrng,2) ];
    [U1L,S1L] = svd(uv4L);   [U1R,S1R] = svd(uv4R);
    SingVals1L = diag(S1L);
    SingVals1R = diag(S1R);
    tx = [ SingVals0(krng)    ;
           SingVals1L([3; 4]) ;
           SingVals1R([3; 4]) ];

    % --- Project the matrices into this very reduced space.
    uvL = U1L(:,[1 2]);    uvR = U1R(:,[1 2]);
    Kred = uvL.'*K0*uvR;   Dred = uvL.'*D0*uvR;    Mred = uvL.'*M0*uvR;

    % --- Determine an improved estimate of an "eigenvalue" now.
    CM = [ Z2  I2;  (-Mred\Kred)  (-Mred\Dred) ];  rts2 = eig(CM);
    pair0 = [ ktt; dtt; mtt ];          % Store existing <approx> {k,d,m}.
    pair0 = pair0 / sqrt(pair0.'*pair0);    % Normalise <pair0>.
    pair1 = real([(rts2(1)*rts2(2)); -(rts2(1)+rts2(2)); 1 ]);
    pair1 = pair1 / sqrt(pair1.'*pair1);    % Normalise <pair1>.
    pair2 = real([(rts2(3)*rts2(4)); -(rts2(3)+rts2(4)); 1 ]);
    pair2 = pair2 / sqrt(pair2.'*pair2);    % Normalise <pair2>.
    tz = abs([ (pair0'*pair1)  (pair0'*pair2)]);
    if (tz(1) > tz(2)), pair3=pair1; else pair3=pair2; end

    % --- Now "update" the "eigenvalue".
    pair_new = pair3 / sqrt(pair3'*pair3);
    disp(pair_new');
    ktt = pair_new(1);       dtt = pair_new(2);       mtt = pair_new(3);

    istep = istep+1;         ty = tx.'*tx;
    disp([' Step # ' int2str(istep) ': vector residuals = ' num2str(ty) ]);
    pause(0.1);
    if (ty < 1.0E-25), igo=0; end

end

disp(' * * * ALGORITHM complete * * * ');
disp(' The {k,d,m} values extracted are (/100) : ');
disp([ num2str(ktt*100) ' : ' num2str(dtt*100) ' : ' num2str(mtt*100) ])
```