QUADRATIC EIGENVALUE PROBLEMS: SOLVENTS AND INVERSE PROBLEMS

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Early Gohberg/Lancaster/Rodman Theory

U. of Calgary, Dept of Math and Stat, Research report 419, 1979. (126pp)

Papers in LAA, Canadian J.Math., IEOT, SIMAX, LAMA. 1977-1982.

Gohberg, I., Lancaster, P., and Rodman, L., *Spectral analysis of matrix polynomials*, Annals of Math, 1982.

Gohberg, I., Lancaster, P., and Rodman, L., *Matrix Polynomials*, Academic Press, 1982. (SIAM 2009 - to appear)

The quadratic problem

Let M, D, $K \in \mathbb{C}^{n\times n}$, all Hermitian, with $M > 0$ and

$$L(\lambda) := M\lambda^2 + D\lambda + K.$$

We will assume that L is monic; $M = I$.

We will also consider the important special case of D and K real and symmetric.

Factorization of $L(\lambda)$

In general, if $D^* = D$ and $K^* = K$, do there exist matrices A (a right divisor) and S (a left divisor) such that

$$I_n\lambda^2 + D\lambda + K = (I_n\lambda - S)(I_n\lambda - A)?$$

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YES! But, existence theorem "evenly divides" eigenvalues with respect to the real axis.

There are many more options for the right and left divisors.
We take advantage of factorizations to solve an inverse problem as follows:

- Assign n eigenvalues and eigenvectors as those of a right divisor A.
- Determine a class of compatible left divisors S.
- Choose a suitable S

Preliminaries

Suppose that there are $2r$ real evs and $2(n-r)$ non-real ev (in $n-r$ conjugate pairs).

A right divisor immediately determines n of these ev's and n associated right eigenvectors, because

$$(I\lambda - A)x = 0, \quad \Rightarrow \quad L(\lambda)x = (I_n\lambda - S)(I_n\lambda - A)x = 0.$$

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THE TRIVIAL SOLUTION: We can solve the Hermitian problem by taking $S = A^*$. We eschew this solution! Why?

Because real eigenvalues of $(I_n\lambda - A^*)(I_n\lambda - A)$ are necessarily defective and

$$(I_n\lambda - A^*)(I_n\lambda - A) \geq 0 \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

The Hermitian case

Let $S = S_R + iS_I$ where $S_R = \frac{1}{2}(S + S^*)$ and $S_I = \frac{-i}{2}(S - S^*)$ are Hermitian. It is not difficult to see that, for a left solvent S,

$$S_R A - A^* S_R = \frac{1}{2}(A^2 - (A^*)^2). \tag{1}$$

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Theorem

Given a matrix $A \in \mathbb{C}^{n\times n}$ a matrix $S \in \mathbb{C}^{n\times n}$ is such that both $S + A$ and SA are Hermitian if and only if

$$S = S_R - \frac{1}{2}(A - A^*) = S_R - iA_I,$$

where S_R is an Hermitian solution of (1).

When the proposition holds we have, of course,

$$D = -(S + A) = -(S_R + A_R), \qquad K = SA.$$

The real-symmetric case

With $A \in \mathbb{R}^{n \times n}$ write

$$A = A_1 + A_2 \quad \text{where} \quad A_1 = \frac{1}{2}(A + A^T), \quad A_2 = \frac{1}{2}(A - A^T),$$

and similarly for S.

Corollary

Given a matrix $A \in \mathbb{R}^{n \times n}$ a matrix $S \in \mathbb{R}^{n \times n}$ is such that both $S + A$ and SA are real and symmetric if and only if $S_2 = -A_2$ and S_1 is a real symmetric solution of

$$S_1 A - A^T S_1 = \frac{1}{2}(A^2 - (A^T)^2). \tag{2}$$

Real systems in real arithmetic

A is now to be a real right divisor and we seek real left divisors S. For the spectrum of A write

$$\sigma(A) = \{\sigma_1, \dots, \sigma_{2r}\} \cup \{\rho_1, \dots, \rho_s\} \tag{3}$$

where $2r + s = n$ and

- $\sigma_1 = \mu_1 + i\omega_1$, $\sigma_2 = \mu_2 - i\omega_2, \dots, \sigma_{2r} = \mu_{2r} - i\omega_{2r}$, are distinct non-real numbers.
- $\rho_1, \dots, \rho_s$ are distinct real numbers.

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- $\rho_1, \ldots, \rho_s$ are distinct real numbers.

Consider a real Jordan canonical form, J_R, for A, assuming that all eigenvalues are simple:

$$J_R = \text{diag}\left[\begin{bmatrix} \mu_1 & \omega_1 \\ -\omega_1 & \mu_1 \end{bmatrix}, \ldots \begin{bmatrix} \mu_r & \omega_r \\ -\omega_r & \mu_r \end{bmatrix}, \rho_1, \ldots, \rho_s\right], \tag{4}$$

and there is real (e-vector) matrix B such that $A = BJ_RB^{-1}$.

Real systems in real arithmetic

Lemma

If A has n distinct eigenvalues (as in (3)) and $B^{-1}AB = J_R$, then S_1 is a real symmetric solution of $ZA - A^T Z = 0$ if and only if $S_1 = B^{-T} F B^{-1}$ where

$$F = \operatorname{diag}\left[\begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{bmatrix}, \ldots \begin{bmatrix} a_r & b_r \\ b_r & -a_r \end{bmatrix}, c_1, \ldots, c_s\right], \tag{5}$$

and the parameters a_j, b_j, c_k are real.

Notice that $F^T = F$ and $FJ_R = J_R^T F = (FJ_R)^T$.

Real systems in real arithmetic

Theorem

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a real matrix B for which $A = BJ_RB^{-1}$ and (4) holds, all real matrices S for which $A + S$ and AS are real and symmetric have the form $S = S_1 + S_2$ where

$$S_1 = A_1 + B^{-T}FB^{-1}, \qquad S_2 = -A_2 \tag{6}$$

and F is an arbitrary real symmetric matrix with the structure of (5).

Real systems in real arithmetic

Real symmetric system coefficients generated in this way are:

$$
\begin{aligned}
D &= -(S+A) = -2A_1 - B^{-T}FB^{-1}, && (7)\\
K &= SA = A^TA + B^{-T}(FJ_R)B^{-1}. && (8)
\end{aligned}
$$

Clearly, they are determined by the n real parameters defining F in (5).

In this construction the number of real (or non-real) eigenvalues in A can be doubled (in $L(\lambda)$), and there is no further constraint on their positions in the complex plane.

Real systems in real arithmetic

Example: Assign

$$A = BJ_RB^{-1} = \begin{bmatrix} 2 & -2 & 1 & -4 \\ 2 & -3 & 1 & -3 \\ 2 & -2 & -1 & -2 \\ 4 & -3 & 1 & -5 \end{bmatrix}$$

with eigenvalues $-1 \pm i, \; -2 \; -3$.
If we take

$$F = \mathrm{diag}\left[\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}, -4, \; -4\right],$$

we obtain a matching left divisor

$$S = \begin{bmatrix} -4 & 4 & 1 & -4 \\ 1 & 0 & 1 & -5 \\ 0 & 0 & -9 & 10 \\ 0 & -2 & 13 & -22 \end{bmatrix}$$

with ev -28.9022, 1.5426, -4.8432, -2.7971.

The real symmetric system coefficients obtained are

$$D = \begin{bmatrix} 6 & -3 & -1 & 4 \\ & 0 & -1 & 5 \\ & & 10 & -11 \\ & & & 26 \end{bmatrix}, \quad K = \begin{bmatrix} 7 & -8 & -3 & 13 \\ & 8 & 2 & -8 \\ & & 24 & -41 \\ & & & 99 \end{bmatrix}.$$

The Hermitian case

As we are interested in the case of mixed, real and non-real eigenvalues, suppose that

$$\sigma(A) = \{\sigma_1, \dots, \sigma_r\} \cup \{\rho_1, \dots, \rho_s\} \tag{9}$$

where $r + s = n$ and

- $\sigma_1, \dots, \sigma_r$ are distinct non-real numbers,
- $\sigma_j \neq \bar{\sigma}_k$ for $j, k = 1, \dots, r$ (no conjugate pairs),
- $\rho_1, \dots, \rho_s$ are distinct real numbers.

$$\Delta := \mathrm{diag}(\sigma_1 \cdots \sigma_r), \quad R := \mathrm{diag}(\rho_1 \cdots \rho_s). \tag{10}$$

The Hermitian case

Then A has the spectral decomposition

$$A = \sum_{j=1}^{r} \sigma_j P_j + \sum_{k=1}^{s} \rho_k Q_k, \tag{11}$$

where $P_1, \ldots, P_r, Q_1, \ldots Q_s$ form a mutually biorthogonal system of rank-one projectors onto the eigenspaces of A corresponding to the σ_j, ρ_k, respectively.
(If $r > 0$ then these assumptions do not admit the choice of real matrices A with non-real eigenvalues.)

The Hermitian case

Theorem

Given a matrix $A \in \mathbb{C}^{n\times n}$ for which (11) *and* (10) *hold, all matrices S for which $A+S$ and AS are Hermitian have the form $S = S_R + iS_I$ where, for some diagonal $E \in \mathbb{R}^{s\times s}$,*

$$S_R = \frac{1}{2}(A+A^*) + T^{-*}\begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} T^{-1}, \qquad S_I = \frac{i}{2}(A-A^*). \tag{12}$$

This shows that S is determined by A and the s real parameters defining E.

(If A has **no** real eigenvalues, then the last term in the equation for S_R does not appear and the trivial solution

$$S = \frac{1}{2}(A+A^*) + i.\frac{i}{2}(A-A^*) = A^*$$

is unique.)

Sign charcteristics

In both cases (real/symm. and Hermitian) we can keep track of the sign characteristics of real eigenvalues.

Hermitian systems with $A \in \mathbb{R}^{n \times n}$

Let the right divisor $I\lambda - A$ be defined by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Ev of A are 1, 2, 3.
(a) If we choose $e_1 = e_2 = e_3 = 1$, then

$$S = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix} + \sum_{j=1}^{3} e_j Y_j = \begin{bmatrix} 3 & 5/2 & 3 \\ -1/2 & 7/2 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

$L(\lambda)$ has ev $1, 2, 3$ with sign characteristic -1.

The ev of S (1.78.., 2.69.., 7.02..) interlace those of A and have sign characteristic $+1$.

(b) If $e_1 = e_2 = e_3 = 10$, then

$$S = \begin{bmatrix} 21 & 7 & 11 \\ 4 & 8 & 1 \\ 20 & 2 & 32 \end{bmatrix}$$

with ev 42.83..,12.91.., 5.25.., all of $+$ve type.
This determines a *hyperbolic* system since all ev are real and, assuming the ev's are ordered, $L(\lambda)$ has n consecutive ev's of one type followed by n consecutive ev's of the other type with a gap between the nth and $n+1$st ev's.

(c) If we let $e_1 = e_2 = 1, e_3 = -1$, then

$$S = \begin{bmatrix} 1 & 3/2 & 1 \\ -3/2 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

with evs 1.82.., and a conjugate pair: $2.58.. \pm i(1.05..)$.

EPILOGUE: Self-adjoint Jordan triples

$n \times n$ Hermitian systems: $L(\lambda) = \sum_{j=0}^{l} A_j \lambda^j$, $A_l > 0$.

Spectral properties encapsulated in a self-adjoint Jordan triple:

$$(X,\ J,\ PX^*).$$

X is $n \times ln$ complex (defined by eigenvectors):
J is $ln \times ln$ complex Jordan:
P is $ln \times ln$ real: all entries 0 or ± 1.
(Accounts for sign characteristics of real eigenvalues.)

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$\Rightarrow$ Hermitian moments $\Gamma_j = X(J^j)PX^*$, $j = 0, 1, \ldots$
$\Rightarrow$ formulae for Hermitian coefficients A_j.

Real symmetric systems

Are Hermitian! But have more structure.
Spectral properties encapsulated in a real self-adjoint Jordan triple:

$$(X_R,\ J_R,\ P_R X_R^T).$$

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Time to go!