Polynomial Matrix Decompositions with Engineering Applications

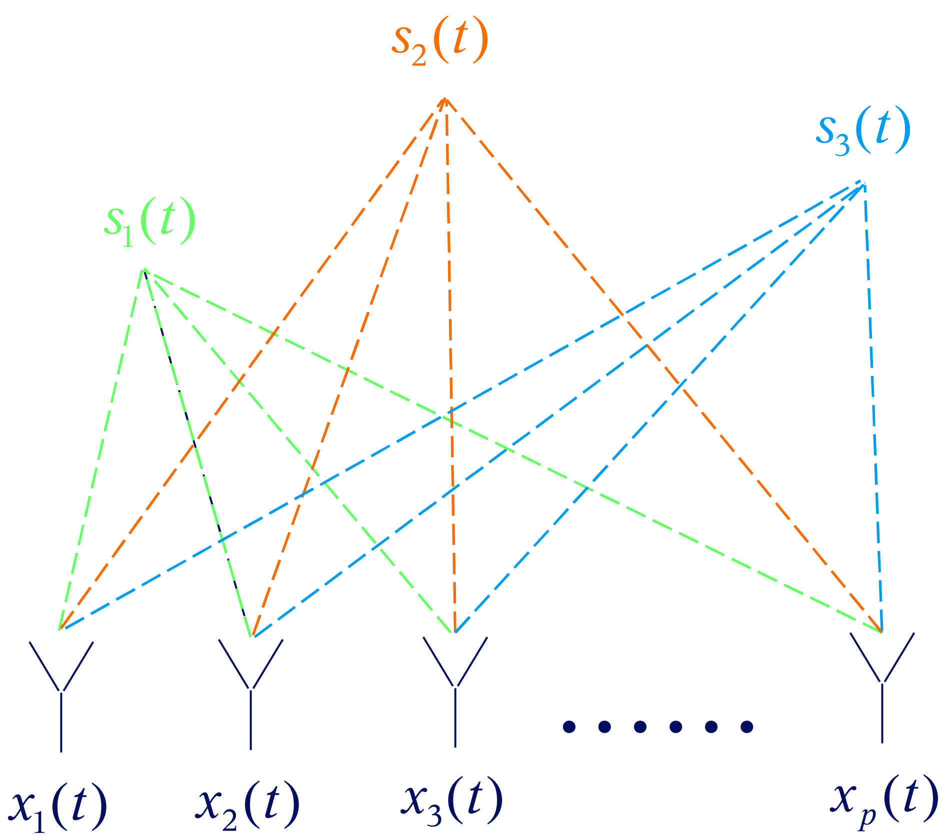
Prof J G McWhirter FRS FREng
School of Engineering
Cardiff University

Dr Joanne Foster MSc PhD
Electronic & Electrical Engineering
Loughborough University

Outline of Talk

- Convolutive mixing and polynomial matrices
  - strong decorrelation by PEVD
- Sequential best rotation algorithm (SBR2)
  - convolutive signal extraction
- Polynomial matrix QR decomposition (PQRD)
  - MIMO channel equalisation

Instantaneous Mixing



- Signal model

$$x(t) = \mathbf{A}\mathbf{s}(t) + n(t)$$

- Data matrix

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{N}$$

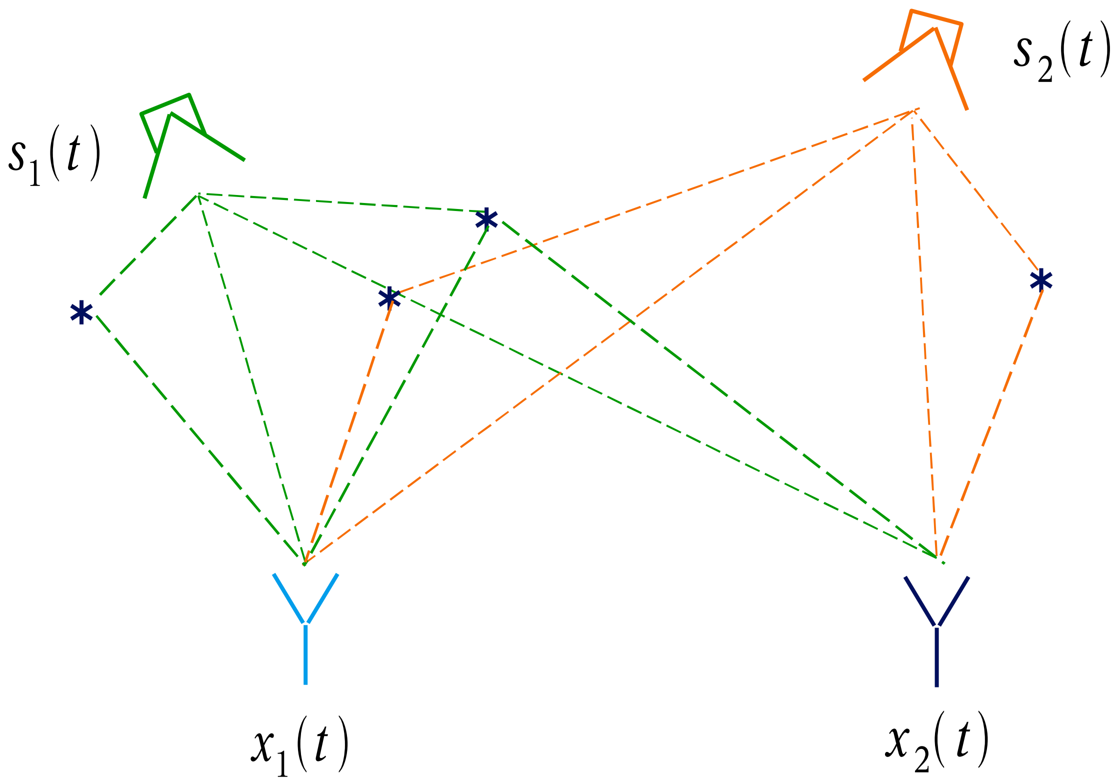
- Sample covariance matrix

$$M = \mathbf{X}\mathbf{X}^{\boldsymbol{H}} \quad m_{ij} = \sum_t x_i(t)\, x_j^{*}(t)$$

- Decorrelation corresponds to EVD of sample covariance matrix

Convolutive Mixing

- Effects of multipath, dispersion etc



Channel Model

- Weighted sum of delayed samples (convolution)

$$x(n) = a_0 s(n) + a_1 s(n-1) + \dots\dots\dots a_p s(n-p)$$

- Express in *polynomial* form (c.f. z-transform)

$$\underline{a}(z) = a_0 + a_1 z^{-1} + \dots\dots\dots a_p z^{-p}$$

$$\underline{s}(z) = s(0) + s(1) z^{-1} + \dots\dots\dots s(n) z^{-n} + \dots\dots\dots$$

$$\underline{x}(z) = x(0) + x(1) z^{-1} + \dots\dots\dots x(n) z^{-n} + \dots\dots\dots$$

- Convolution becomes *simple product*

$$\underline{x}(z) = \underline{a}(z)\underline{s}(z)$$

- Underscore used to denote polynomial quantities

CARDIFF UNIVERSITY PRIFYSGOL CAERDYDD

Polynomial Matrix Formulation

- Two signals and two sensors

$$\underline{x}_1(z)=\underline{a}_{11}(z)\underline{s}_1(z)+\underline{a}_{12}(z)\underline{s}_2(z)$$
$$\underline{x}_2(z)=\underline{a}_{21}(z)\underline{s}_1(z)+\underline{a}_{22}(z)\underline{s}_2(z)$$

- Polynomial matrix form

$$\begin{bmatrix}\underline{x}_1(z)\\ \underline{x}_2(z)\end{bmatrix}=\begin{bmatrix}\underline{a}_{11}(z) & \underline{a}_{12}(z)\\ \underline{a}_{21}(z) & \underline{a}_{22}(z)\end{bmatrix}\begin{bmatrix}\underline{s}_1(z)\\ \underline{s}_2(z)\end{bmatrix}$$

- i.e

$$\underline{x}(z)=\underline{A}(z)\underline{s}(z)$$

Special Polynomial Matrices

- Unimodular Matrix $\det[A(z)] = const$

- Paraconjugation $\tilde{A}(z) = A_{\text{¿}}^{T}(1/z)$

- Para-Hermitian Matrix $\tilde{A}(z) = A(z)$

- Paraunitary matrix (defines multichannel all-pass filter)

$$H(z)\tilde{H}(z) = \tilde{H}(z)H(z) = I$$

Polynomial Matrix Examples

$$\underline{A}(z)=\begin{bmatrix} 1+z^{-1} & z+2 \\ z^{-1} & 2 \end{bmatrix}$$

- Paraconjugate

$$\tilde{\underline{A}}(z)=\begin{bmatrix} 1+z & z \\ z^{-1}+2 & 2 \end{bmatrix}$$

- Inverse

$$A^{-1}(z)=\begin{bmatrix} 2 & -(z+2) \\ -z^{-1} & 1+z^{-1} \end{bmatrix} \qquad \det\left[A(z)\right]=1$$

- Holds for all values of z - time domain processing

Strong Decorrelation by Paraunitary Matrix

- Paraunitary transformation $\underline{H}(z)\underline{x}(z)=\underline{v}(z)$

  hence $R_{vv}(z)=\underline{H}(z)R_{xx}(z)\tilde{\underline{H}}(z)$ where $R_{xx}(z)=\sum_{\tau} R_{xx}(\tau)z^{-\tau}$

  and $\left[R_{xx}(\tau)\right]_{ij}=E\{x_i(t)x_j(t-\tau)\}$

- Design $\underline{H}(z)$ to achieve strong decorrelation i.e. $\left[R_{vv}(\tau)\right]_{ij}=E\{v_i(t)v_j(t-\tau)\}=d_i(\tau)\delta_{ij}$

- or $\underline{R}_{vv}(z)=\begin{bmatrix} \underline{d}_1(z) & 0 \\ 0 & \underline{d}_p(z) \end{bmatrix}$ where $\underline{d}_i(z)=\sum_{\tau} d_i(\tau)z^{-\tau}$

Polynomial Matrix EVD

- Corresponds to diagonalising cross spectral density matrix (Laurent polynomial) by paraunitary similarity transform

$$\underline{R}_{vv}(z) = \underline{H}(z)\underline{R}_{xx}(z)\underline{\tilde{H}}(z) = \begin{bmatrix} \underline{d}_1(z) & 0 \\ 0 & \underline{d}_p(z) \end{bmatrix} \quad (z = e^{j\omega} \ \forall \ \omega)$$

- Taken as definition of polynomial matrix EVD (PEVD)
- For polynomials of order zero ⟶ scalar EVD

Elementary Paraunitary Transformation

- Applies integer delay followed by scalar rotation

$$\underline{\Gamma}(z)=\begin{bmatrix} c & s \\ -s^{¿} & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z^{-N} \end{bmatrix}$$

- Note that

$$\underline{\Gamma}(z)\,\tilde{\underline{\Gamma}}(z)=\begin{bmatrix} c & s \\ -s^{¿} & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z^{-N} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z^{N} \end{bmatrix}\begin{bmatrix} c & -s \\ s^{¿} & c \end{bmatrix}=I$$

Elementary Paraunitary Operation

- Covariance domain

$$\underline{\Gamma}(z)\begin{bmatrix}\underline{r}_{ii}(z) & \underline{r}_{ij}(z)\\ \underline{r}_{ji}(z) & \underline{r}_{jj}(z)\end{bmatrix}\underline{\tilde{\Gamma}}(z)=\begin{bmatrix}\underline{r}''_{ii}(z) & \underline{r}''_{ij}(z)\\ \underline{r}''_{ji}(z) & \underline{r}''_{jj}(z)\end{bmatrix}$$

where

$$\underline{r}_{ij}(z)=\sum_{\tau} r_{ij}(\tau)z^{-\tau} \qquad r_{ij}(\tau)=E\{x_i(t)x_j^{*}(t-\tau)\}$$

and

$$r''_{ij}(0)=0=r''_{ji}(0)$$

SBR2 Algorithm

- Given the estimated space-time covariance matrix

$$r_{ij}(\tau)=\sum_{t=1}^{T} x_i(t)\,x_j(t-\tau)/T$$

- Locate dominant cross-correlation coefficient $r_{kl}(t)$
- Shift to zero-lag position by applying relative delay operator
- Apply rotation to achieve decorrelation at zero lag

$$\tan(2\theta)=2|r_{kl}(0)|/(r_{kk}(0)-r_{ll}(0))$$

- Drives $r'_{kl}(0)$ to zero and updates covariance matrix
- Elementary paraunitary operation is repeated iteratively

SBR2 Algorithm

- Initial Polynomial covariance matrix

$$\begin{bmatrix} r_{11}(-\tau) & r_{12}(-\tau) & r_{13}(-\tau) \\ r_{21}(-\tau) & r_{22}(-\tau) & r_{23}(-\tau) \\ r_{31}(-\tau) & r_{32}(-\tau) & r_{33}(-\tau) \end{bmatrix}$$

$$\begin{bmatrix} r_{11}(0) & r_{12}(0) & r_{13}(0) \\ r_{21}(0) & r_{22}(0) & r_{23}(0) \\ r_{31}(0) & r_{32}(0) & r_{33}(0) \end{bmatrix}$$

$$\begin{bmatrix} r_{11}(\tau) & r_{12}(\tau) & r_{13}(\tau) \\ r_{21}(\tau) & r_{22}(\tau) & r_{23}(\tau) \\ r_{31}(\tau) & r_{32}(\tau) & r_{33}(\tau) \end{bmatrix}$$

SBR2 Algorithm

- Single stage (assuming dominant coefficient is $r_{13}(1)$)
  - modifies every plane in polynomial matrix
  - order of polynomial grows (zero padding)

$$\begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s^{¿} & 0 & c \end{bmatrix}$$

$$\begin{bmatrix} r_{11}(\tau) & r_{12}(\tau) & r_{13}(\tau+1) \\ r_{21}(\tau) & r_{22}(\tau) & r_{23}(\tau+1) \\ r_{31}(\tau-1) & r_{32}(\tau-1) & r_{33}(\tau) \end{bmatrix}$$

$$\begin{bmatrix} r_{11}(0) & r_{12}(0) & r_{13}(1) \\ r_{21}(0) & r_{22}(0) & r_{23}(1) \\ r_{31}(-1) & r_{32}(-1) & r_{33}(0) \end{bmatrix}$$

$$\begin{bmatrix} r_{11}(-\tau) & r_{12}(-\tau) & r_{13}(1-\tau) \\ r_{21}(-\tau) & r_{22}(-\tau) & r_{23}(1-\tau) \\ r_{31}(-1-\tau) & r_{32}(-1-\tau) & r_{33}(-\tau) \end{bmatrix}$$

$$\begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s^{¿} & 0 & c \end{bmatrix}$$

Cost Function for SBR2

- Zero-lag measures

$$N_1 = \sum_i |r_{ii}(0)|^2$$

invariant to delay

$$N_2 = \sum_i \sum_j |r_{ij}(0)|^2$$

invariant to rotation

$$N_3 = N_2 - N_1 = \sum_i \sum_{j \neq i} |r_{ij}(0)|^2$$

- Strong decorrelation measure

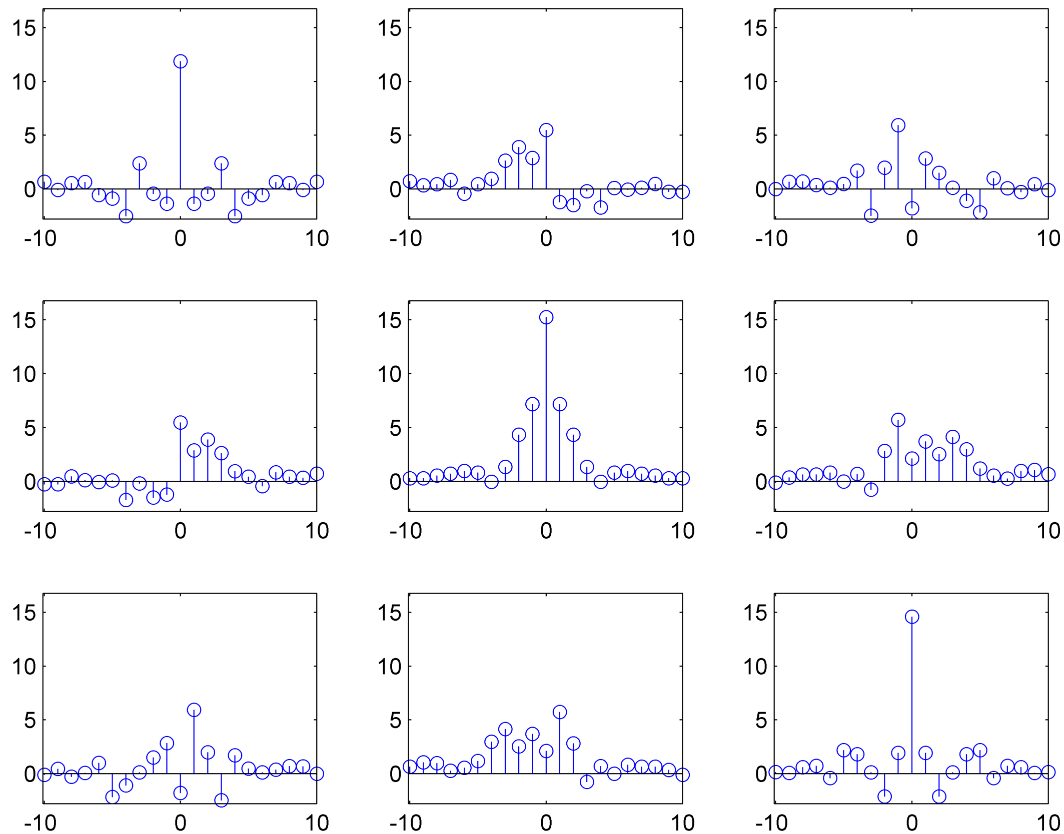
$$M = \max_{i,j,\tau} \{|r_{ij}(\tau)|\} \quad (i \neq j)$$

- Proven convergence

$$M \to 0$$

Initial Polynomial Covariance Matrix

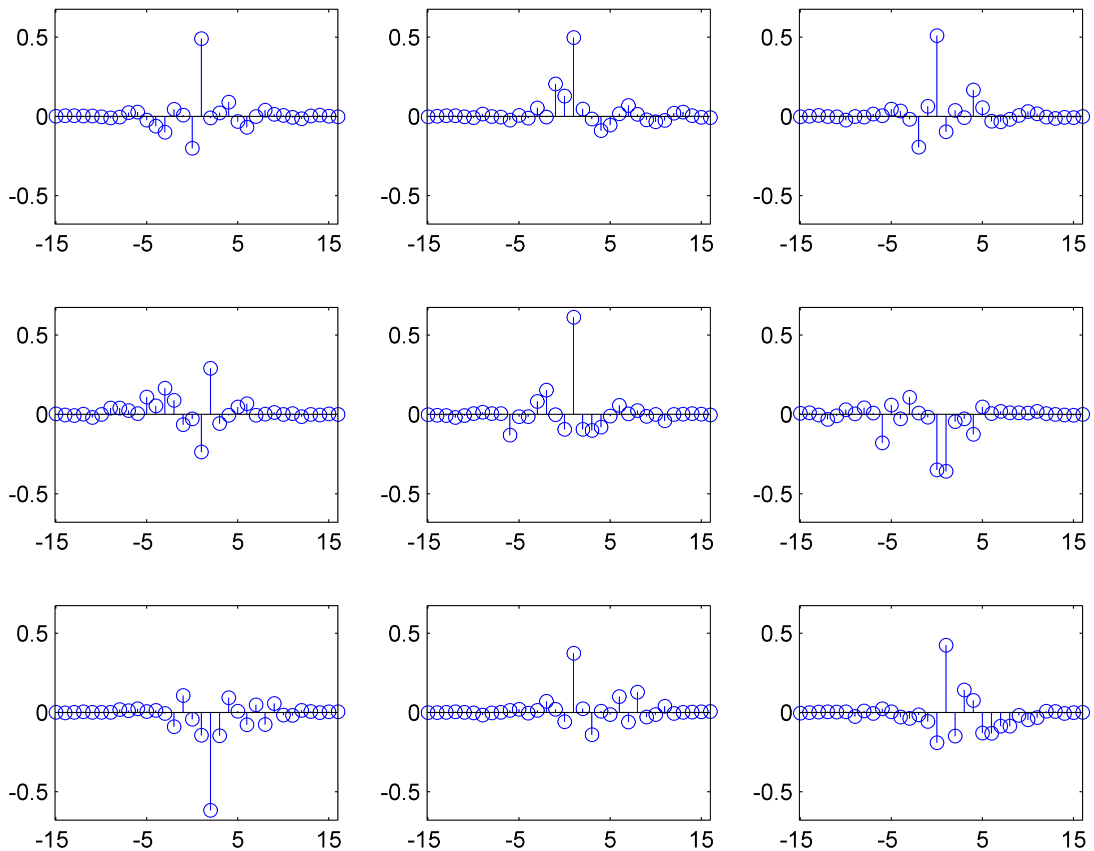
$$R_{xx}(z)$$



2 sources, 3 sensors, order 5 mixing filters, coefficients random in [-1 1]

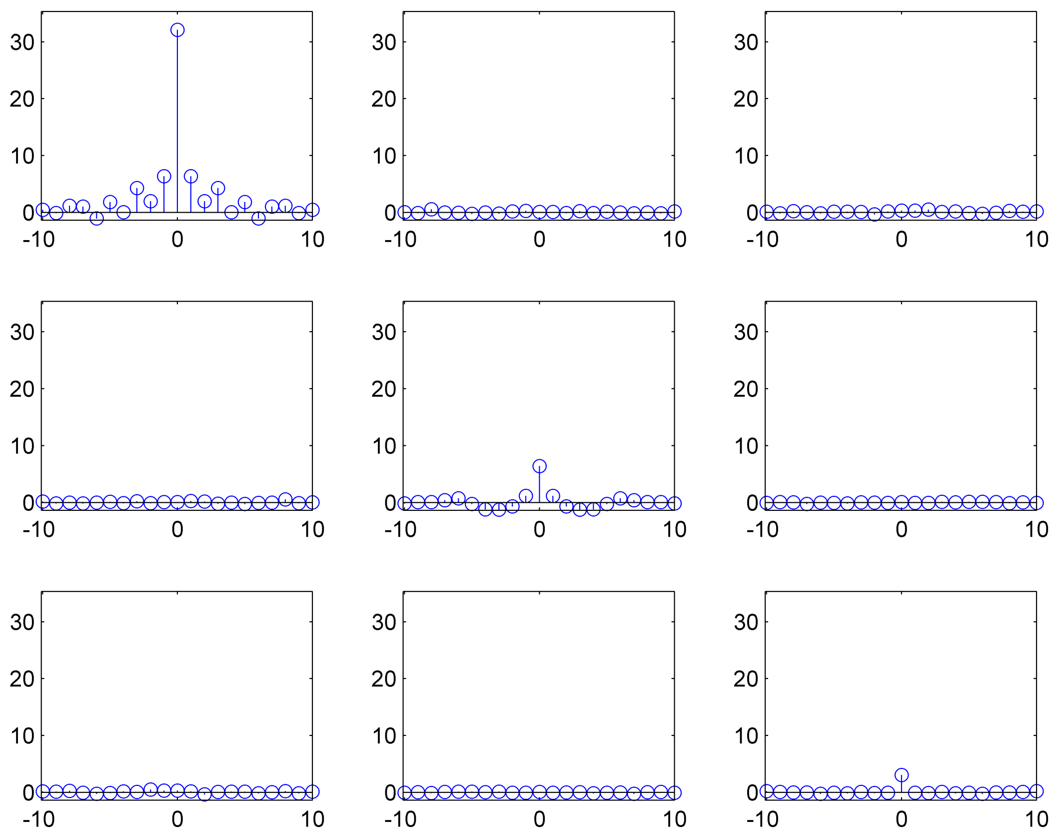
Paraunitary Transformation Matrix

$$H(z)$$

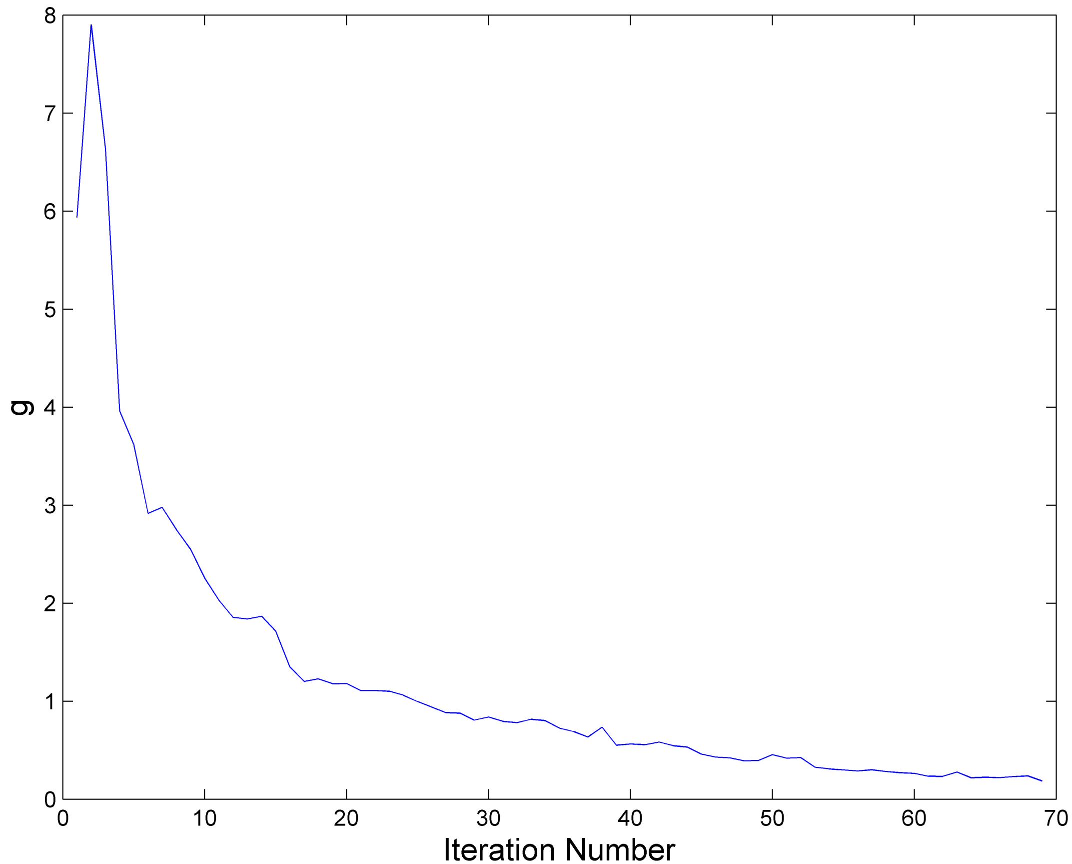


Effect of Strong Decorrelation

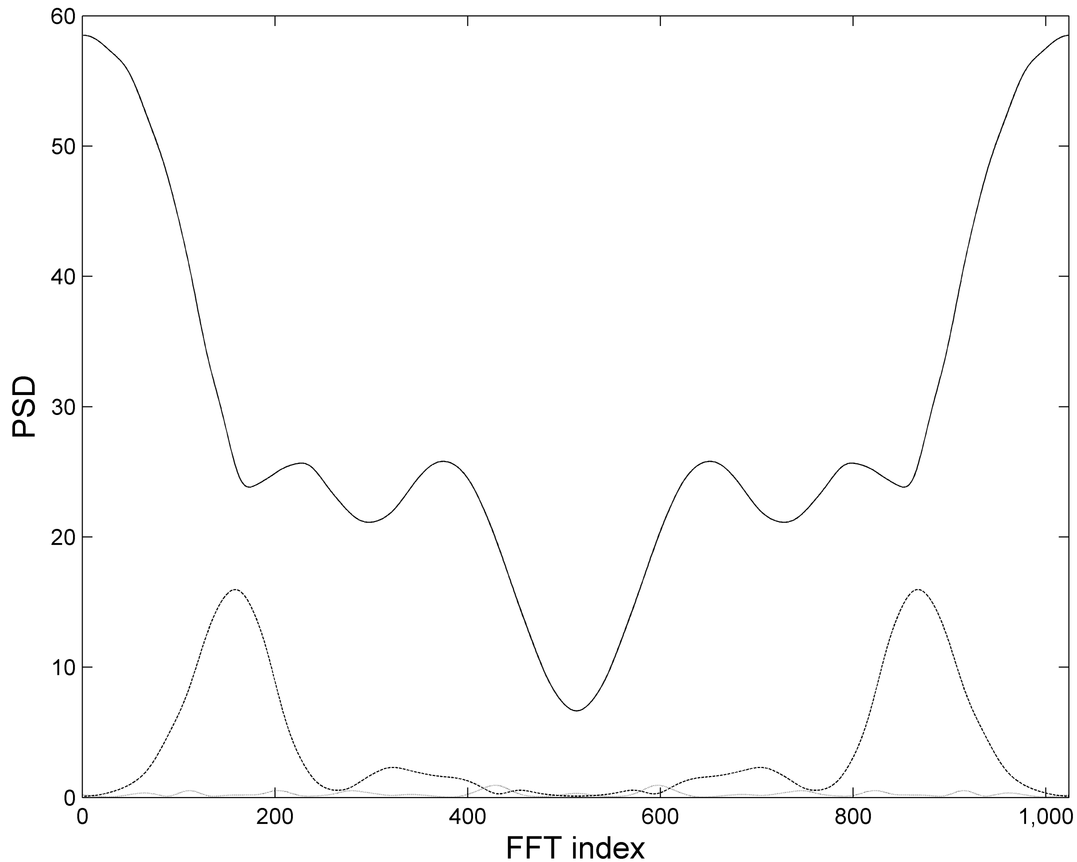
$R_{vv}(z)$



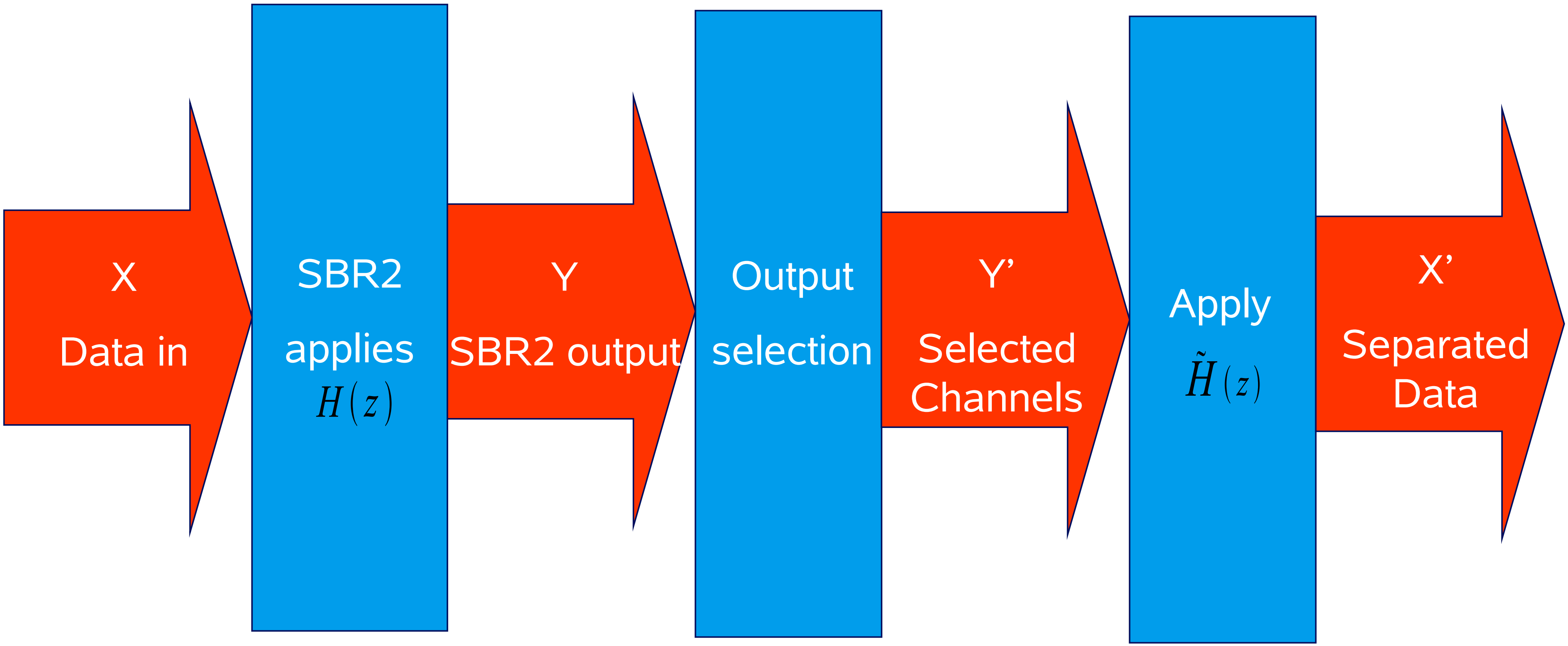
Convergence of SBR2 Algorithm



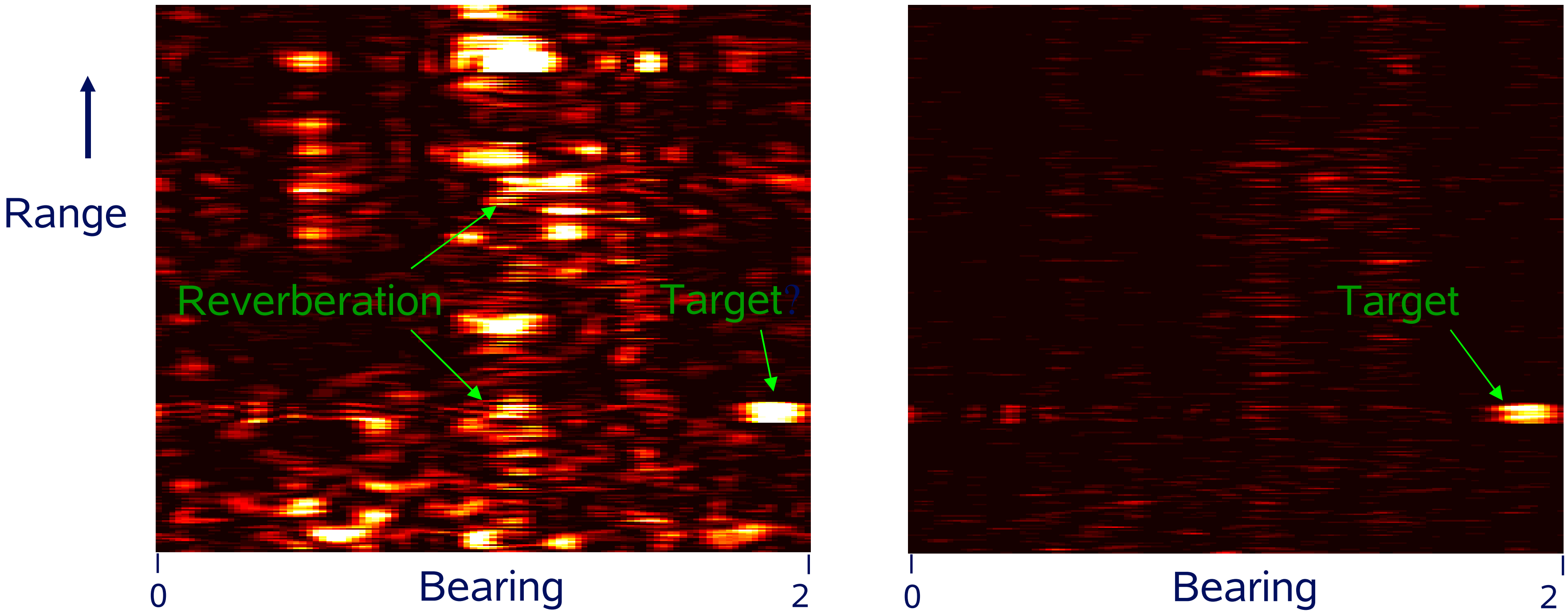
Spectral Majorisation of Output Signals



Signal separation using SBR2



Sonar reverberation suppression



- The SBR2 algorithm can been used to reduce reverberation considerably
- A target is revealed at a bearing of ~343°

QR Decomposition of a Polynomial Matrix

Input matrix $\underline{A}(z) \in \underline{C}^{p \times q}$

Objective is to calculate $\underline{Q}(z) \in \underline{C}^{p \times p}$ such that

$$\underline{A}(z) = \underline{Q}(z)\underline{R}(z)$$

where $\underline{R}(z) \in \underline{C}^{p \times q}$ is upper triangular and $\underline{Q}(z)$ is paraunitary, i.e.

$$\underline{\tilde{Q}}(z)\underline{Q}(z) = \underline{Q}(z)\underline{\tilde{Q}}(z) = I$$

Complete Polynomial Givens Rotation

CARDIFF UNIVERSITY PRIFYSGOL CAERDYDD

$$\underline{G}(z)\begin{bmatrix}\underline{a}_1(z)\\ \underline{a}_2(z)\end{bmatrix}\simeq\begin{bmatrix}\underline{a}'_1(z)\\ 0\end{bmatrix}$$

- Series of EPGR's $\underline{G}(z)=\underline{\hat{G}}_i(z)\ldots\ldots\underline{\hat{G}}_1(z)$
- This matrix is paraunitary $\underline{\tilde{G}}(z)\underline{G}(z)=\underline{G}(z)\underline{\tilde{G}}(z)=I$
- All coefficients of $\underline{a}_2(z)$ have been driven arbitrarily close to zero

$$|a_2(\hat{t})|<\varepsilon$$

MIMO Channel Equalisation

Cardiff University / Prifysgol Caerdydd

Convolutive Mixing Model $$x(t)=\sum_{k=0}^{N} A(k)s(t-k)+n(t)$$

Or expressed in polynomial form $$\underline{x}(z)=\underline{A}(z)\underline{s}(z)+\underline{n}(z)$$

Assume channel matrix is known, then $$\underline{A}(z)=\underline{Q}(z)\underline{R}(z)$$

Rearranging $$\underbrace{\underline{\tilde{Q}}(z)\underline{x}(z)}_{\underline{x}'(z)}=\underline{R}(z)\underline{s}(z)+\underbrace{\underline{\tilde{Q}}(z)\underline{n}(z)}_{\underline{n}'(z)}$$

2x2 Example

CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

$$\begin{bmatrix} \underline{x}_1'(z) \\ \underline{x}_2'(z) \end{bmatrix} = \begin{bmatrix} \underline{r}_{11}(z) & \underline{r}_{12}(z) \\ 0 & \underline{r}_{22}(z) \end{bmatrix} \begin{bmatrix} \underline{s}_1(z) \\ \underline{s}_2(z) \end{bmatrix} + \begin{bmatrix} \underline{n}_1'(z) \\ \underline{n}_2'(z) \end{bmatrix}$$

Use **back substitution** to...

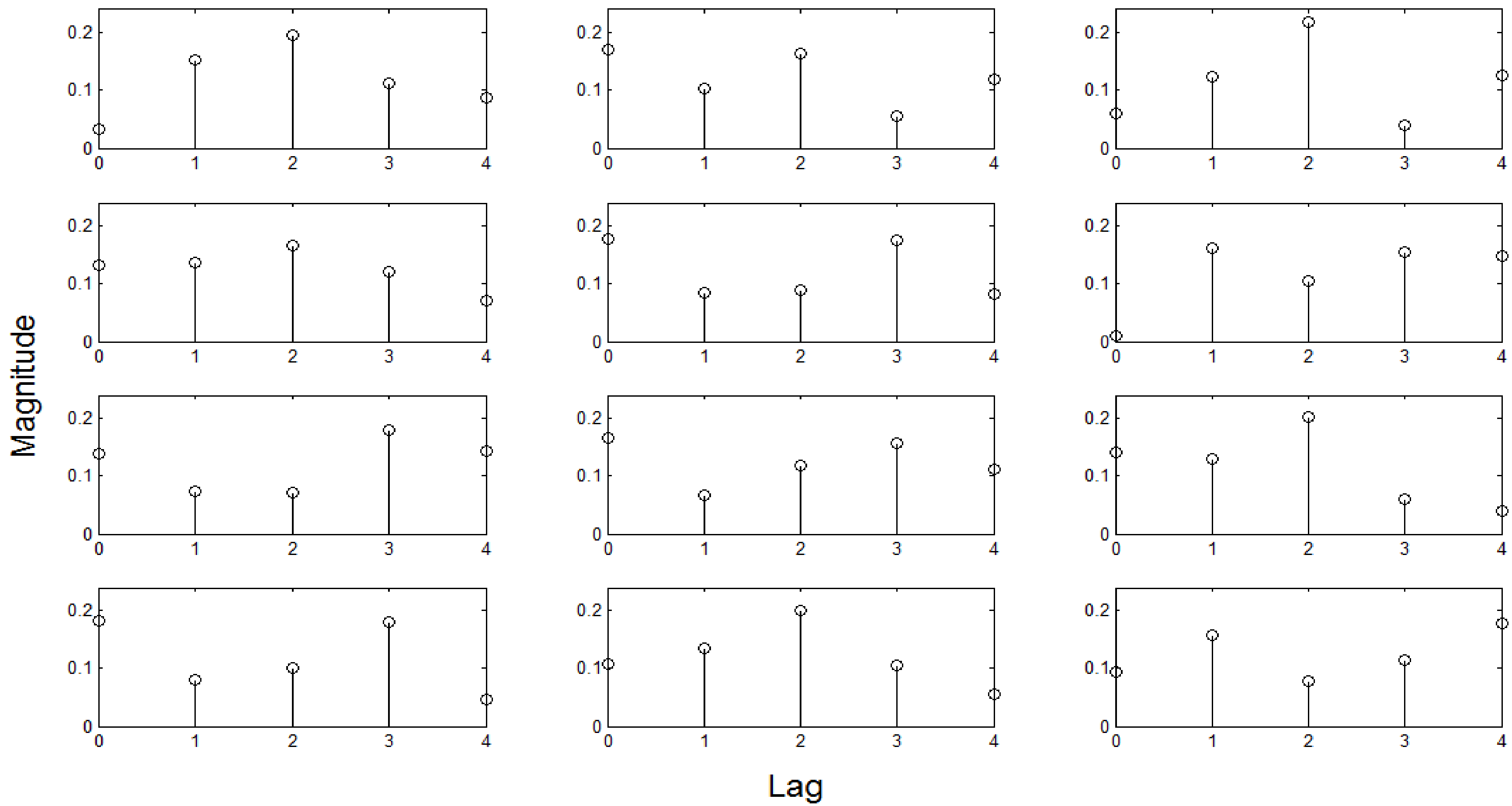
1. Estimate source 2 $\underline{x}_2'(z) = \underline{r}_{22}(z)\,\underline{s}_2(z) + \underline{n}_2'(z)$

Single channel equalisation problem – solve using a maximum likelihood sequence estimator

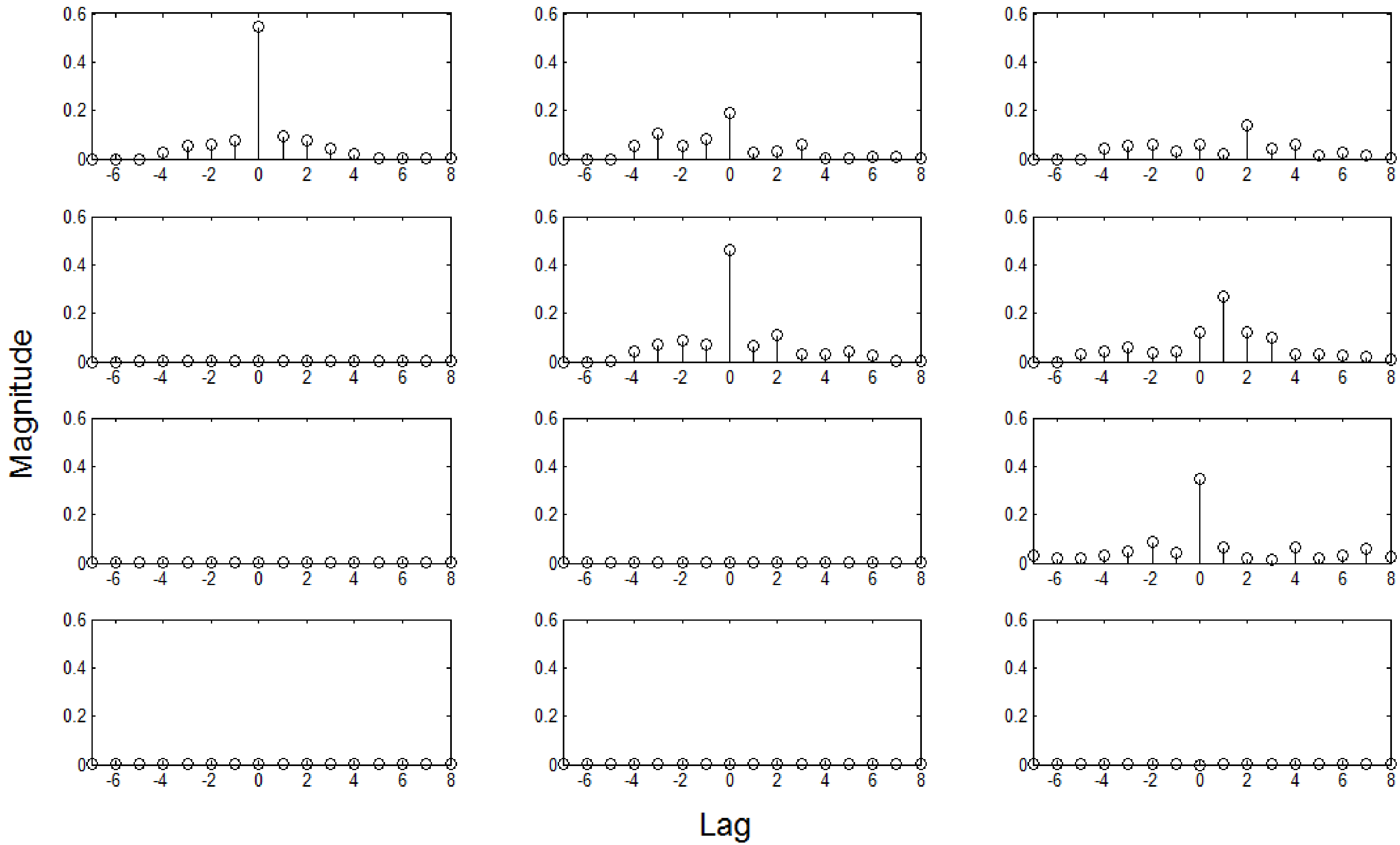
2. Estimate source 1 $\underline{x}_1'(z) - \underline{r}_{12}(z)\,\underline{s}_2(z) = \underline{r}_{11}(z)\,\underline{s}_1(z) + \underline{n}_1'(z)$

Simple Example

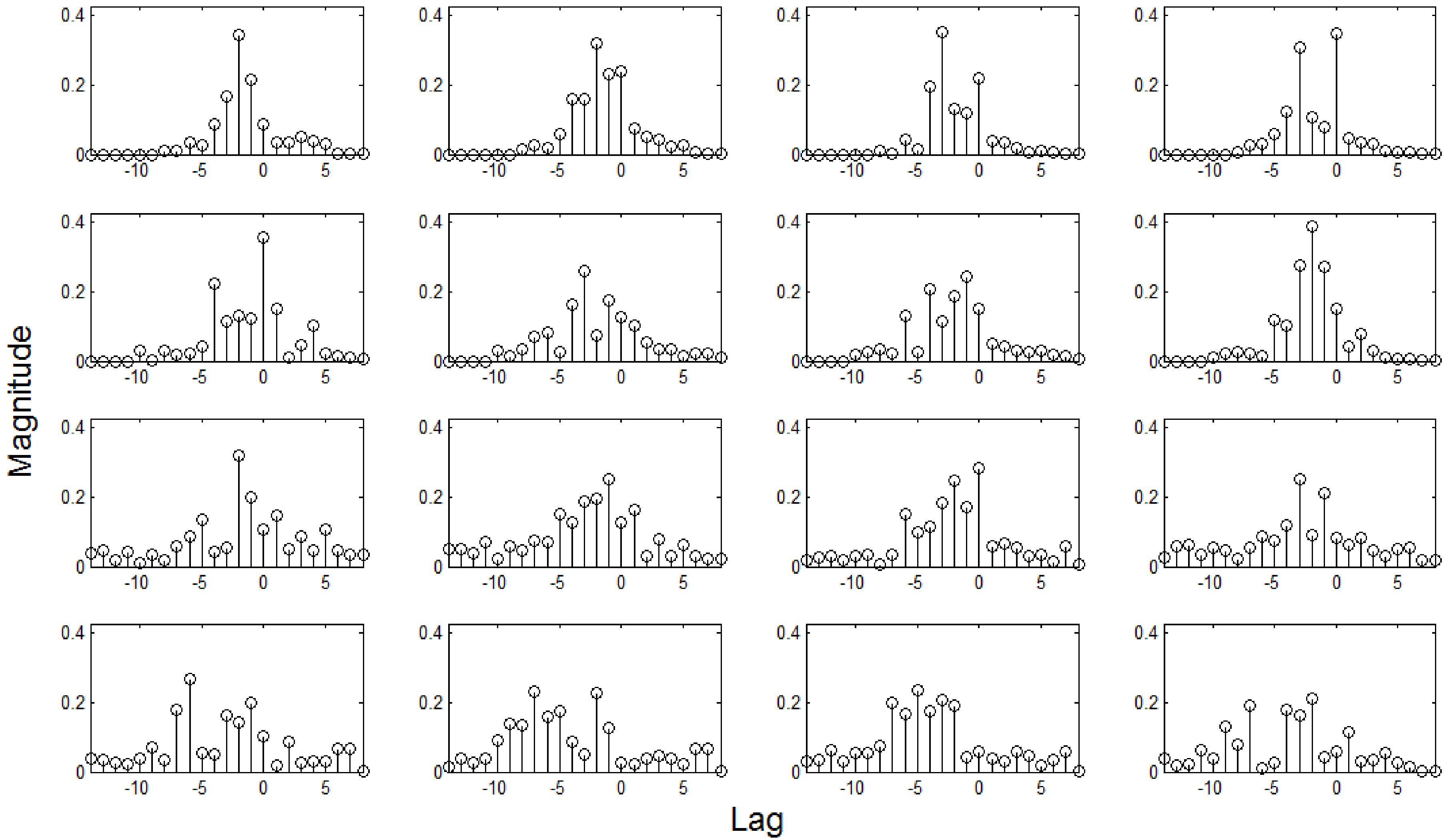
Input Matrix $\underline{\mathbf{A}}(z)$



Upper Triangular Matrix R(z)



Paraunitary Matrix **Q**(z)



Average BER for 3 Source Signals at Varying Noise Levels

RSNR (dB)	Average BER		
	Source 1	Source 2	Source 3
-5	0.1914	0.1938	0.2693
0	0.0644	0.0729	0.1470
5	0.0032	0.0056	0.0319
10	0	0	0.0007
15	0	0	0
20	0	0	0

Acknowledgements

- This research was sponsored by the United Kingdom Ministry of Defence, Corporate Research Programme.
- Reference
  - J G McWhirter, P D Baxter, T Cooper, S Redif and J Foster

    "An EVD Algorithm for Para-Hermitian Polynomial Matrices"

    IEEE Trans Signal Processing, Vol 55, No 6 (May 2007).
- Thanks to co-authors from QinetiQ and Cardiff University whose work is also presented here.