Pencils, Spectra and Pseudospectra

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Pencils of Operators

The spectrum of a family of bounded operators $A(z)$ acting from a Banach space $\mathcal{B}_1$ to $\mathcal{B}_2$ is defined by

$$\mathrm{Spec}(A(\cdot)) = \{z : A(z) \text{ is not invertible}\}.$$

This reduces to the normal definition if $\mathcal{B}_1 = \mathcal{B}_2$ and $A(z) = zI - B$.

Matrix Pencils

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If also

$$A(z) = \sum_{r=0}^{m} A_r z^r$$

then $\mathrm{Spec}(A(\cdot))$ is finite.

Pseudospectra of Pencils

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In a Hilbert space context one may alternatively assume that the smallest singular value of $A(z)$ is less than ε.

Approximation Procedures

If P_n is an increasing sequence of orthogonal projections on $\mathcal{H}$, one may try to approximate $\mathrm{Spec}(B)$ by $\mathrm{Spec}(P_n B P_n)$. This frequently fails even when B is self-adjoint.

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$$A(z) = P_n(\overline{z}I - B^*)P_n(zI - B)P_n + P_n B^*(I - P_n)BP_n.$$

Let us now assume that B is self-adjoint. There are now two options, both carried out within $P_n\mathcal{H}$. [1]

- Replace $\overline{z}$ by z and determine the spectrum of the quadratic pencil

$$P_n(zI - B)^2 P_n$$

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  for complex z;
- Determine the pseudospectra of

$$P_n(xI - B)^2 P_n$$

  for real x. In other words calculate the smallest eigenvalue of this non-negative operator as a function of $x \in \mathbf{R}$.

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Approximate Eigenfunctions

If one is interested in the eigenvectors as well as the eigenvalues of a non-self-adjoint operator A then a more complicated framework is needed. If $H : X \to \mathbf{C}$ is a classical Hamiltonian on a phase space X one may try to find $\psi \in \mathcal{H}$ such that $\|\psi\| = 1$, ψ is supported near x and $\|A\psi - H(x)\psi\|$ is small simultaneously.[2]

[2] See E B Davies: J. Diff. Eqns. 216 (2005) 153-187.

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If $X = R^{2n}$ and $(x, \xi) \in X$ and $\mathcal{H} = L^2(\mathbf{R}^n)$ this would be achieved if

$$\begin{aligned} \|Q\psi - x\psi\| &< \varepsilon \\ \|P\psi - \xi\psi\| &< \varepsilon \\ \|A\psi - H(x,\xi)\psi\| &< \varepsilon. \end{aligned}$$

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This is impossible unless one introduces h and thinks about the semi-classical limit $h \to 0+$.

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This can be rewritten as a pseudospectral pencil problem for operators from $L^2(\mathbf{R}^n)$ to $L^2(R^n) \otimes \mathbf{C}^3$, namely

$$\|L(x,\xi)\psi\| < \varepsilon \|\psi\|$$

where

$$L(x,\xi)\psi = (Q\psi - x\psi, P\psi - \xi\psi, A\psi - H(x,\xi)\psi).$$

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In applications one should not assume that values of $H(x,\xi)$ for which this is soluble are close to eigenvalues of the original operator A.

Semi-classical Analysis

We consider

$$(L_h f)(x) = -h^2 a_h^{j,k}(x)\partial_{j,k} f(x) - ihb_h^j(x)\partial_j f(x) + c_h(x)f(x)$$

acting on functions $f : \mathbf{R}^N \to \mathbf{C}$, where a, b, c are sufficiently regular functions whose values are respectively matrices, vectors and scalars with complex-valued entries.

We assume that the coefficients have finite limits as $h \to 0+$.

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The semi-classical principal symbol of this is

$$H(u, \xi) = a_0^{j,k}(u)\xi_j \xi_k + b_0^j(u)\xi_j + c_0(u)$$

Invariance of the Symbol

The following alternative definition of the semiclassical principal symbol of L_h makes its invariant character clear. Suppose that $u \in X$ and ξ is a cotangent vector at u. Let f be any smooth function on X such that $\mathrm{d}f(u) = \xi$. Then

$$H(u,\xi) = \left\{ \lim_{h \to 0} \mathrm{e}^{-ih^{-1}f} L_h \left(\mathrm{e}^{ih^{-1}f} \right) \right\} (u).$$

Theorem

Suppose that $a_h^{j,k}(x)$, $b_h^j(x)$ *and* $c_h(x)$ *are all locally Lipschitz continuous in both* $x \in \mathbf{R}^N$ *and* $h \in [0,1]$. *Then for every* $u \in \mathbf{R}^N$, $\xi \in \mathbf{R}^N$ *and* $h \in (0,1]$ *there exists* $\psi_h \in C_c^\infty(\mathbf{R}^N)$ *such that*

$$\begin{aligned} \|\psi_h\|_2 &= c > 0 \\ \|Q\psi_h - u\psi_h\|_2 &= O(h^{1/2}) \\ \|P\psi_h - \xi\psi_h\|_2 &= O(h^{1/2}) \\ \|L_h\psi_h - H(x,\xi)\psi_h\|_2 &= O(h^{1/2}) \end{aligned}$$

as $h \to 0$, *where*

$$\begin{aligned} (Q\psi)(x) &= x\psi(x) \\ (P\psi)(x) &= -ih\nabla\psi(x). \end{aligned}$$

Boundary Pseudospectra

If $\mathbf{R}^n$ is replaced by a bounded region U with smooth boundary, one can construct approximate solutions for all $x \in U$ and ξ in the tangent space at x. However, there is another class of approximate solutions, each one concentrated around some $x \in \partial U$.

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For each $x \in U$ the family of solutions is parametrized by those complexified cotangent vectors that are orthogonal to the boundary tangents and have positive real parts in all interior directions.

The definitions and theorems are invariant under coordinate changes.