On the Class of Isospectral Systems

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Notations & Definitions 1(3)

$A \in M_N(\mathbb{C})$ Eigenvalues and Eigenvectors

$$Aq = \lambda q \Longleftrightarrow (\lambda I_N - A)q = 0$$

Jordan Canonical Form

$$AQ = QJ$$

Jordan Matrix $J = \mathrm{diag}(J_1, \cdots, J_h)$,

$J_i = \mathrm{diag}(J_{i,1}, \cdots, J_{i,k_i})$

$$J_{i,\nu} = \begin{bmatrix} \lambda_i & 1 & 0 & & & \\ 0 & \lambda_i & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda_i & 1 & 0 \\ & & & 0 & \lambda_i & 1 \\ & & & & 0 & \lambda_i \end{bmatrix} \in M_{n_{i,\nu}}(\mathbb{C})$$

Segre Characteristic of eigenvalue λ_i

$$(n_{i,1}, \cdots, n_{i,k_i})$$

Notations & Definitions 2(3)

Matrix Polynomial

$$L(\lambda) = \lambda^{\ell} A_{\ell} + \cdots + \lambda A_1 + A_0,\ A_i \in M_n(\mathbb{C})$$

- Degree $\deg(L(\lambda)) = \ell$
- Dimension $\dim(L(\lambda)) = n$
- Regular $:\Leftrightarrow \det(L(\lambda)) \not\equiv 0$
- Unimodular $:\Leftrightarrow \det(L(\lambda)) = c \neq 0$
- Monic $:\Leftrightarrow A_{\ell} = I_n$

Notations & Definitions 3(3)

General Matrix Polynomial

$$L(\lambda) = \lambda^\ell A_\ell + \cdots + \lambda A_1 + A_0,$$

here the A_i are $m \times n$.

There exist unimodular Matrix Polynomials $U(\lambda) \in M_m(\mathbb{C}), V(\lambda) \in M_n(\mathbb{C})$ such that

$$L(\lambda) = U(\lambda) D(\lambda) V(\lambda)$$

Smith Normal Form

$$D(\lambda) = \mathrm{diag}(d_1(\lambda), \cdots, d_r(\lambda), 0, \cdots, 0)$$

$d_i(\lambda)$ invariant factors or elementary divisors satisfying $d_{i-1}(\lambda) | d_i(\lambda)$, trivial $d_k(\lambda) = 1$

$\mathcal{E}(L(\lambda)) := (d_1(\lambda), \cdots, d_p(\lambda))$ nontrivial elementary divisors

Linearizations 1(2)

Linear Matrix Polynomial

$$\lambda A + B$$

unimodular Matrix Polynomials $E(\lambda), G(\lambda)$

$$E(\lambda)(\lambda A + B)G(\lambda) = \text{diag}(I_n, \cdots, I_n, L(\lambda))$$

then $\lambda A + B$ is a Linearization for $L(\lambda)$

Monic case:

$L(\lambda) = \lambda^\ell I_n + \lambda^{\ell-1} A_{\ell-1} + \cdots + \lambda A_1 + A_0$

Companion Linearization $\lambda I_{n\ell} - C$

Companion Matrix

$$C = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_n \\ -A_0 & -A_1 & \cdots & -A_{\ell-2} & -A_{\ell-1} \end{bmatrix}$$

Linearizations 2(2)

Unimodular Matrix Polynomials for $\lambda I_{n\ell} - C$

$$G(\lambda) := \begin{bmatrix} 0 & \cdots & 0 & I_n \\ \vdots & \iddots & \iddots & \lambda I_n \\ 0 & \iddots & \iddots & \vdots \\ I_n & \lambda I_n & \cdots & \lambda^{\ell-1} I_n \end{bmatrix}$$

$$E(\lambda) := \begin{bmatrix} 0 & \cdots & 0 & -I_n & 0 \\ \vdots & \iddots & \iddots & 0 & \vdots \\ 0 & \iddots & \iddots & \vdots & \vdots \\ -I_n & 0 & \cdots & 0 & 0 \\ P_{\ell-1}(\lambda) & \cdots & \cdots & P_1(\lambda) & I_n \end{bmatrix}$$

where, for $i = 1, \cdots, \ell - 1$,

$$P_i(\lambda) := \sum_{k=0}^{i-1} A_{\ell+k-i} \lambda^k + \lambda^i I_n$$

Isospectral Systems

Matrix Polynomials $L(\lambda)$ and $\widetilde{L}(\lambda)$ are called

isospectral $:\Leftrightarrow$

they share the same (nontrivial) elementary divisors $d_i(\lambda)$.

$$\mathcal{C}_D := \{L(\lambda) \,|\, \mathcal{E}(L(\lambda)) = (d_1(\lambda), \cdots, d_p(\lambda))\}$$

- Equivalence Transformations
- Generate $\mathcal{C}_D$
- Overview over $\mathcal{C}_D$
- Canonical Form $L_c(\lambda) \in \mathcal{C}_D$

Structure Preserving Transformations 1(4)

$L(\lambda)$ monic and $Y \in M_n(\mathbb{C})$ nonsingular
$\rightsquigarrow \widetilde{L}(\lambda) := YL(\lambda)Y^{-1}$ and $L(\lambda)$ are isospectral.

Are there more?

Monic case $\lambda I_{n\ell} - C$. For any $X \in \mathbb{C}^{n \times n\ell}$ such that

$$T_n := \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{\ell-1} \end{bmatrix}$$

is nonsingular

$$T_n C T_n^{-1} = \widetilde{C}$$

is a Companion Matrix associated with

$$\widetilde{L}(\lambda) = \lambda^\ell I_n - XC^\ell T_n^{-1} \begin{bmatrix} I_n \\ \lambda I_n \\ \vdots \\ \lambda^{\ell-1} I_n \end{bmatrix}$$

T_n is called
Structure Preserving Similarity (SPS) for C.

Structure Preserving Transformations 2(4)

Special cases:

$Y \in M_n(\mathbb{C})$ nonsingular, $X = [Y\, 0 \cdots 0]$ then $T_n = I_n \otimes Y$ and

$$\widetilde{L}(\lambda) = \lambda^\ell I_n - XC^\ell T_n^{-1} \begin{bmatrix} I_n \\ \lambda I_n \\ \vdots \\ \lambda^{\ell-1} I_n \end{bmatrix} = Y L(\lambda) Y^{-1}$$

strict similarity transformation of $L(\lambda)$.

$X = [I_n\, 0 \cdots 0]$ then $T_n = I_{n\ell}$ and $T_n C T_n^{-1} = C$

More general consider the Centralizer

$$\mathcal{Z}_C := \{Z \mid ZC = CZ\}$$

then every nonsingular $T \in \mathcal{Z}_C$ is called an Automorphic SPS for C.

Structure Preserving Transformations 3(4)

$$\begin{tikzcd}
L(\lambda) \arrow[d, "\mathcal{L}"'] & & \widetilde{L}(\lambda) \\
\lambda I_{n\ell} - C \arrow[rr, "\Theta_X"] & & \lambda I_{n\ell} - \widetilde{C} \arrow[u, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

Given $P(\lambda)$ of degree ℓ and dimension n with Jordan Matrix J. Define

$$\mathbb{S}_n := \{X \in \mathbb{C}^{n \times n\ell} \,|\, T_n \text{ nonsingular}\}$$

then

$$\{(\mathcal{L}^{-1}\Theta_X\mathcal{L})L(\lambda) \,|\, X \in \mathbb{S}_n\} \subseteq \mathcal{C}_D$$

Are there more?

Structure Preserving Transformations 4(4)

Let $m, k \in \mathbb{N}, k > 1$, such that $mk = n\ell$. For any $X \in \mathbb{C}^{m \times n\ell}$ with

$$T_m := \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{k-1} \end{bmatrix} \text{ nonsingular}$$

$T_m C T_m^{-1} = \widetilde{C}$ is a companion matrix associated with the monic system

$$\widetilde{L}(\lambda) = \lambda^k I_m - XC^k T_m^{-1} \begin{bmatrix} I_m \\ \lambda I_m \\ \vdots \\ \lambda^{k-1} I_m \end{bmatrix}$$

of dimension m and degree k.

$$\{(\mathcal{L}^{-1} \ominus_X \mathcal{L}) L(\lambda) \,|\, X \in \mathbb{S}_m, m \neq n\ell, m | (n\ell)\} = \mathcal{C}_D$$

Examples 1(3)

Several cases for the initial system

$$L(\lambda) =$$

$$\lambda^3 I_2 - \frac{\lambda^2}{7}\begin{bmatrix} 43 & 1 \\ 328 & 27 \end{bmatrix} + \frac{\lambda}{7}\begin{bmatrix} 69 & 6 \\ 960 & 141 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 57 & 1 \\ 888 & 209 \end{bmatrix}$$

with $\dim(L(\lambda)) = 2$ and $\deg(L(\lambda)) = 3$

Case A: $(m, k) = (1, 6)$

$X_1 = (1, 2, 3, 4, 5, 6) \in \mathbb{S}_1$ with $\mathrm{cond}(T_1) \approx 10^4$ and $T_1 C T_1^{-1} = \widetilde{C}$ companion matrix associated with the one-dimensional system of degree 6

$$\widetilde{L}(\lambda) = \lambda^6 - 10\lambda^5 + 47\lambda^4 - 140\lambda^3 + 271\lambda^2 - 330\lambda + 225$$

the characteristic polynomial of C.

Examples 2(3)

Case B: $(m,k) = (2,3)$

For

$$X_2 = \begin{bmatrix} 44 & -40 & 0 & 0 & 0 & 0 \\ 204 & 127 & -292 & -26 & 0 & 7 \end{bmatrix}$$

T_2 has condition number $\approx$ 90 and applying the SPS yields the upper-triangular system

$$\widetilde{L}(\lambda) =$$

$$\lambda^3 I_2 - \lambda^2 \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix} + \lambda \begin{bmatrix} 11 & 10 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 15 & 11 \\ 0 & 15 \end{bmatrix}$$

Examples 3(3)

Case C: $(m, k) = (3, 2)$

The choice

$$X_3 = \begin{bmatrix} 39 & 67 & -102 & -32 & 7 & 7 \\ 12 & 166 & -312 & -60 & 28 & 14 \\ 861 & 273 & -630 & -42 & 21 & 21 \end{bmatrix}$$

leads to cond$(T_3) \approx 386$ and $T_3 C T_3^{-1} = \widetilde{C}$ is the companion matrix of the monic, bi-diagonal system of degree 2

$$\widetilde{L}(\lambda) = \lambda^2 I_3 - \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Filters 1(2)

$$
\begin{array}{ccc}
L(\lambda) & \xrightarrow{\quad\mathcal{F}\quad} & \widetilde{L}(\lambda) \\
\mathcal{L}\Big\downarrow & & \Big\uparrow\mathcal{L}^{-1} \\
\lambda I_{n\ell} - C & \xrightarrow{\quad\Theta_X\quad} & \lambda I_{n\ell} - \widetilde{C}
\end{array}
$$

- $E(\lambda)(\lambda I_{n\ell} - C)G(\lambda) = \mathrm{diag}(I_n, \cdots, I_n, L(\lambda))$

- $\widetilde{E}(\lambda)(I_{n\ell} - \widetilde{C})\widetilde{G}(\lambda) = \mathrm{diag}(I_m, \cdots, I_m, \widetilde{L}(\lambda))$

- $T_m(I_{n\ell} - C)T_m^{-1} = I_{n\ell} - \widetilde{C}$

Filters 2(2)

Filters $F(\lambda)$ and $\widetilde{F}(\lambda)$ isospectral Matrix Polynomials of degree $\ell-1, k-1$, resp.

$$\widetilde{F}(\lambda)L(\lambda) = \widetilde{L}(\lambda)F(\lambda)$$

Example:

$$L(\lambda) = \lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F(\lambda) = \begin{bmatrix} 1 & 2\lambda-1 \\ 1 & -\lambda-1 \end{bmatrix}, \widetilde{F}(\lambda) = \begin{bmatrix} \lambda-1 & 2\lambda+1 \\ 1 & -1 \end{bmatrix}$$

$$\widetilde{F}(\lambda)L(\lambda)(F(\lambda))^{-1} = \mathrm{diag}(\lambda^2+\lambda+1, \lambda-1)$$

Conclusions

- Isospectral systems can have different degree and dimension.

- Explicit knowledge on spectral properties is not required.

- Isospectral systems are related via structure preserving transformations.

- The concept of Filters does not require any linearization.

References

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