

On the Class of Isospectral Systems

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Notations & Definitions 1(3)

$A \in M_N(\mathbb{C})$ Eigenvalues and Eigenvectors

$$Aq = \lambda q \iff (\lambda I_N - A)q = 0$$

Jordan Canonical Form

$$AQ = QJ$$

Jordan Matrix $J = \text{diag}(J_1, \dots, J_h)$,

$$J_i = \text{diag}(J_{i,1}, \dots, J_{i,k_i})$$

$$J_{i,\nu} = \begin{bmatrix} \lambda_i & 1 & 0 & & & & \\ 0 & \lambda_i & 1 & 0 & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & \lambda_i & 1 & 0 & \\ & & & 0 & \lambda_i & 1 & \\ & & & & 0 & \lambda_i & \end{bmatrix} \in M_{n_{i,\nu}}(\mathbb{C})$$

Segre Characteristic of eigenvalue λ_i

$$(n_{i,1}, \dots, n_{i,k_i})$$

Notations & Definitions 2(3)

Matrix Polynomial

$$L(\lambda) = \lambda^\ell A_\ell + \cdots + \lambda A_1 + A_0, \quad A_i \in M_n(\mathbb{C})$$

- Degree $\deg(L(\lambda)) = \ell$
- Dimension $\dim(L(\lambda)) = n$
- Regular $:\Leftrightarrow \det(L(\lambda)) \neq 0$
- Unimodular $:\Leftrightarrow \det(L(\lambda)) = c \neq 0$
- Monic $:\Leftrightarrow A_\ell = I_n$

Notations & Definitions 3(3)

General Matrix Polynomial

$$L(\lambda) = \lambda^\ell A_\ell + \cdots + \lambda A_1 + A_0,$$

here the A_i are $m \times n$.

There exist unimodular Matrix Polynomials $U(\lambda) \in M_m(\mathbb{C})$, $V(\lambda) \in M_n(\mathbb{C})$ such that

$$L(\lambda) = U(\lambda)D(\lambda)V(\lambda)$$

Smith Normal Form

$$D(\lambda) = \text{diag}(d_1(\lambda), \cdots, d_r(\lambda), 0, \cdots, 0)$$

$d_i(\lambda)$ invariant factors or elementary divisors
satisfying $d_{i-1}(\lambda) | d_i(\lambda)$, trivial $d_k(\lambda) = 1$

$\mathcal{E}(L(\lambda)) := (d_1(\lambda), \cdots, d_p(\lambda))$ nontrivial elementary divisors

Linearizations 1(2)

Linear Matrix Polynomial

$$\lambda A + B$$

unimodular Matrix Polynomials $E(\lambda), G(\lambda)$

$$E(\lambda)(\lambda A + B)G(\lambda) = \text{diag}(I_n, \dots, I_n, L(\lambda))$$

then $\lambda A + B$ is a **Linearization** for $L(\lambda)$

Monic case:

$$L(\lambda) = \lambda^\ell I_n + \lambda^{\ell-1} A_{\ell-1} + \dots + \lambda A_1 + A_0$$

Companion Linearization $\lambda I_{n\ell} - C$

Companion Matrix

$$C = \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & I_n \\ -A_0 & -A_1 & \dots & -A_{\ell-2} & -A_{\ell-1} \end{bmatrix}$$

Linearizations 2(2)

Unimodular Matrix Polynomials for $\lambda I_{n\ell} - C$

$$G(\lambda) := \begin{bmatrix} 0 & \cdots & 0 & I_n \\ \vdots & \ddots & \ddots & \lambda I_n \\ 0 & \ddots & \ddots & \vdots \\ I_n & \lambda I_n & \cdots & \lambda^{\ell-1} I_n \end{bmatrix}$$

$$E(\lambda) := \begin{bmatrix} 0 & \cdots & 0 & -I_n & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ -I_n & 0 & \cdots & 0 & 0 \\ P_{\ell-1}(\lambda) & \cdots & \cdots & P_1(\lambda) & I_n \end{bmatrix}$$

where, for $i = 1, \dots, \ell - 1$,

$$P_i(\lambda) := \sum_{k=0}^{i-1} A_{\ell+k-i} \lambda^k + \lambda^i I_n$$

Isospectral Systems

Matrix Polynomials $L(\lambda)$ and $\tilde{L}(\lambda)$ are called **isospectral** : \Leftrightarrow

they share the same (nontrivial) elementary divisors $d_i(\lambda)$.

$$\mathcal{C}_D := \{L(\lambda) \mid \mathcal{E}(L(\lambda)) = (d_1(\lambda), \dots, d_p(\lambda))\}$$

- Equivalence Transformations
- Generate \mathcal{C}_D
- Overview over \mathcal{C}_D
- Canonical Form $L_c(\lambda) \in \mathcal{C}_D$

Structure Preserving Transformations 1(4)

$L(\lambda)$ monic and $Y \in M_n(\mathbb{C})$ nonsingular
 $\rightsquigarrow \tilde{L}(\lambda) := YL(\lambda)Y^{-1}$ and $L(\lambda)$ are isospectral.

Are there more?

Monic case $\lambda I_{nl} - C$. For any $X \in \mathbb{C}^{n \times nl}$ such that

$$T_n := \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{\ell-1} \end{bmatrix}$$

is nonsingular

$$T_n C T_n^{-1} = \tilde{C}$$

is a Companion Matrix associated with

$$\tilde{L}(\lambda) = \lambda^\ell I_n - X C^\ell T_n^{-1} \begin{bmatrix} I_n \\ \lambda I_n \\ \vdots \\ \lambda^{\ell-1} I_n \end{bmatrix}$$

T_n is called

Structure Preserving Similarity (SPS) for C .

Structure Preserving Transformations 2(4)

Special cases:

$Y \in M_n(\mathbb{C})$ nonsingular, $X = [Y \ 0 \ \dots \ 0]$ then
 $T_n = I_n \otimes Y$ and

$$\tilde{L}(\lambda) = \lambda^\ell I_n - XC^\ell T_n^{-1} \begin{bmatrix} I_n \\ \lambda I_n \\ \vdots \\ \lambda^{\ell-1} I_n \end{bmatrix} = YL(\lambda)Y^{-1}$$

strict similarity transformation of $L(\lambda)$.

$X = [I_n \ 0 \ \dots \ 0]$ then $T_n = I_{n\ell}$ and $T_n C T_n^{-1} = C$

More general consider the Centralizer

$$\mathcal{Z}_C := \{Z \mid ZC = CZ\}$$

then every nonsingular $T \in \mathcal{Z}_C$ is called
an **Automorphic SPS** for C .

Structure Preserving Transformations 3(4)

$$\begin{array}{ccc}
 L(\lambda) & & \tilde{L}(\lambda) \\
 \mathcal{L} \downarrow & & \uparrow \mathcal{L}^{-1} \\
 \lambda I_{nl} - C & \xrightarrow{\Theta_X} & \lambda I_{nl} - \tilde{C}
 \end{array}$$

Given $P(\lambda)$ of degree ℓ and dimension n with Jordan Matrix J . Define

$$\mathbb{S}_n := \{X \in \mathbb{C}^{n \times nl} \mid T_n \text{ nonsingular}\}$$

then

$$\{(\mathcal{L}^{-1}\Theta_X\mathcal{L})L(\lambda) \mid X \in \mathbb{S}_n\} \subseteq \mathcal{C}_D$$

Are there more?

Structure Preserving Transformations 4(4)

Let $m, k \in \mathbb{N}, k > 1$, such that $mk = nl$. For any $X \in \mathbb{C}^{m \times nl}$ with

$$T_m := \begin{bmatrix} X \\ XC \\ \vdots \\ XC^{k-1} \end{bmatrix} \text{ nonsingular}$$

$T_m C T_m^{-1} = \tilde{C}$ is a companion matrix associated with the monic system

$$\tilde{L}(\lambda) = \lambda^k I_m - X C^k T_m^{-1} \begin{bmatrix} I_m \\ \lambda I_m \\ \vdots \\ \lambda^{k-1} I_m \end{bmatrix}$$

of dimension m and degree k .

$$\{(\mathcal{L}^{-1} \Theta_X \mathcal{L}) L(\lambda) \mid X \in \mathbb{S}_m, m \neq nl, m \mid (nl)\} = \mathcal{C}_D$$

Examples 1(3)

Several cases for the initial system

$$L(\lambda) = \lambda^3 I_2 - \frac{\lambda^2}{7} \begin{bmatrix} 43 & 1 \\ 328 & 27 \end{bmatrix} + \frac{\lambda}{7} \begin{bmatrix} 69 & 6 \\ 960 & 141 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 57 & 1 \\ 888 & 209 \end{bmatrix}$$

with $\dim(L(\lambda)) = 2$ and $\deg(L(\lambda)) = 3$

Case A: $(m, k) = (1, 6)$

$X_1 = (1, 2, 3, 4, 5, 6) \in \mathbb{S}_1$ with $\text{cond}(T_1) \approx 10^4$
and $T_1 C T_1^{-1} = \tilde{C}$ companion matrix associated
with the one-dimensional system of degree 6

$$\tilde{L}(\lambda) = \lambda^6 - 10\lambda^5 + 47\lambda^4 - 140\lambda^3 + 271\lambda^2 - 330\lambda + 225$$

the characteristic polynomial of C .

Examples 2(3)

Case B: $(m, k) = (2, 3)$

For

$$X_2 = \begin{bmatrix} 44 & -40 & 0 & 0 & 0 & 0 \\ 204 & 127 & -292 & -26 & 0 & 7 \end{bmatrix}$$

T_2 has condition number ≈ 90 and applying the SPS yields the upper-triangular system

$$\tilde{L}(\lambda) = \lambda^3 I_2 - \lambda^2 \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix} + \lambda \begin{bmatrix} 11 & 10 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 15 & 11 \\ 0 & 15 \end{bmatrix}$$

Examples 3(3)

Case C: $(m, k) = (3, 2)$

The choice

$$X_3 = \begin{bmatrix} 39 & 67 & -102 & -32 & 7 & 7 \\ 12 & 166 & -312 & -60 & 28 & 14 \\ 861 & 273 & -630 & -42 & 21 & 21 \end{bmatrix}$$

leads to $\text{cond}(T_3) \approx 386$ and $T_3 C T_3^{-1} = \tilde{C}$ is the companion matrix of the monic, bi-diagonal system of degree 2

$$\tilde{L}(\lambda) = \lambda^2 I_3 - \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Filters 1(2)

$$\begin{array}{ccc}
 L(\lambda) & \xrightarrow{\mathcal{F}} & \tilde{L}(\lambda) \\
 \mathcal{L} \downarrow & & \uparrow \mathcal{L}^{-1} \\
 \lambda I_{nl} - C & \xrightarrow{\Theta_X} & \lambda I_{nl} - \tilde{C}
 \end{array}$$

- $E(\lambda)(\lambda I_{nl} - C)G(\lambda) = \text{diag}(I_n, \dots, I_n, L(\lambda))$
- $\tilde{E}(\lambda)(I_{nl} - \tilde{C})\tilde{G}(\lambda) = \text{diag}(I_m, \dots, I_m, \tilde{L}(\lambda))$
- $T_m(I_{nl} - C)T_m^{-1} = I_{nl} - \tilde{C}$

Filters 2(2)

Filters $F(\lambda)$ and $\tilde{F}(\lambda)$ **isospectral** Matrix Polynomials of degree $\ell - 1, k - 1$, resp.

$$\tilde{F}(\lambda)L(\lambda) = \tilde{L}(\lambda)F(\lambda)$$

Example:

$$L(\lambda) = \lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F(\lambda) = \begin{bmatrix} 1 & 2\lambda - 1 \\ 1 & -\lambda - 1 \end{bmatrix}, \tilde{F}(\lambda) = \begin{bmatrix} \lambda - 1 & 2\lambda + 1 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{F}(\lambda)L(\lambda)(F(\lambda))^{-1} = \text{diag}(\lambda^2 + \lambda + 1, \lambda - 1)$$

Conclusions

- Isospectral systems can have different degree and dimension.
- Explicit knowledge on spectral properties is not required.
- Isospectral systems are related via structure preserving transformations.
- The concept of Filters does not require any linearization.

References

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