Some open problems in spectral theory

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Approximation of spectra

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Projection methods: Let $(\mathcal{L}_k)_{k\in\mathbb{N}}$ be a sequence of closed linear (finite dimensional) subspaces of $\mathrm{Dom}(A)$ such that the corresponding orthogonal projections $P_k : \mathcal{H} \to \mathcal{L}_k$ converge strongly to the identity operator I. Let $\Lambda(A)$ be the set of all such sequences of subspaces.

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Let $\mathrm{Spec}(A, \mathcal{L}_k)$ be the spectrum of $P_kA : \mathcal{L}_k \to \mathcal{L}_k$. One might hope that

$$\lim_{k\to\infty} \mathrm{Spec}(A, \mathcal{L}_k) = \mathrm{Spec}(A),$$

where "lim" is defined in an appropriate sense.

Example

Let $\mathcal{H} := \ell_2(\mathbb{Z})$,
A be the right shift operator: $Ae_n = e_{n+1}$, $n \in \mathbb{Z}$,
and let $\mathcal{L}_k := \text{span}\{e_n\}_{n=-k}^{k}$.

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Then $\mathrm{Spec}(A) = \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$,

$$\mathrm{Spec}(A, \mathcal{L}_k) = \mathrm{Spec}\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \{0\}.$$

Approximation of spectra

Does

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No!

Approximation of spectra

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This is not the case for unbounded operators.

Theorem. (M. Levitin & ES, 2004)

Let $A = A^*$ be unbounded above . Then for an arbitrary sequence $(\mathcal{L}_k)_{k\in\mathbb{N}} \in \Lambda(A)$ of finite dimensional subspaces and arbitrary $\varepsilon_k \searrow 0$, $R_k \nearrow +\infty$ there exists $(\mathcal{L}'_k)_{k\in\mathbb{N}} \in \Lambda(A)$ such that

$$\|P_k - P'_k\| < \varepsilon_k \quad \textit{and} \quad \mathrm{Spec}(A, \mathcal{L}'_k) \subset (R_k, +\infty), \quad \forall k \in \mathbb{N},$$

where $P'_k : \mathcal{H} \to \mathcal{L}'_k$ are the corresponding orthogonal projections. A similar statement holds for operators unbounded below.

Spectral pollution

Let $\widehat{\mathrm{Spec}}(A)$ the closure of $\mathrm{Spec}(A)$ in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$,

$\mathrm{Spec}_{\mathrm{ess}}(A) := \mathrm{Spec}(A) \backslash \{\text{isolated eigenvalues of finite multiplicity}\}$,

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Theorem. (M. Levitin & ES, 2004; A. Pokrzywa, 1979)

For any $\lambda \in \mathrm{conv}\left(\widehat{\mathrm{Spec}}_{\mathrm{ess}}(A)\right) \setminus \widehat{\mathrm{Spec}}_{\mathrm{ess}}(A)$ *there exists an increasing sequence* $(\mathcal{L}_k)_{k\in\mathbb{N}} \in \Lambda(A)$ *such that*

$$\lambda \in \mathrm{Spec}(A, \mathcal{L}_k), \quad \forall k \in \mathbb{N}.$$

Second order relative spectra of $A = A^*$

E.B. Davies, 1998: Suppose $\mathcal{L} \subset \mathrm{Dom}(A^2)$ and let P be the orthogonal projection onto $\mathcal{L}$. Then

$$\mathrm{Spec}_2(A, \mathcal{L}) := \{z \in \mathbb{C} : \; P(A - zI)^2 : \mathcal{L} \to \mathcal{L} \; \text{ is not invertible}\}.$$

M. Levitin & ES, 2004: Let $\mathcal{L}$ be a finite dimensional subspace of $\mathrm{Dom}(A)$. Then $z \in \mathbb{C}$ is said to belong to $\mathrm{Spec}_2(A, \mathcal{L})$ if there exists $u \in \mathcal{L} \setminus \{0\}$ such that

$$((A - zI)u, (A - \overline{z}I)v) = 0, \quad \forall v \in \mathcal{L}.$$

Second order relative spectra of $A = A^*$

Theorem (ES, 2000; M. Levitin & ES, 2004)

If $z \in \mathrm{Spec}_2(A, \mathcal{L})$ *then*

$$\mathrm{Spec}(A) \cap [\mathrm{Re}\, z - |\mathrm{Im}\, z|, \mathrm{Re}\, z + |\mathrm{Im}\, z|] \neq \emptyset.$$

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No spectral pollution!

Second order relative spectra of $A = A^*$

L. Boulton, 2006:

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Question:

Is it true that

$$\lim_{k\to\infty} \mathrm{Spec}_2(A, \mathcal{L}_k) \supseteq \mathrm{Spec}(A)?$$

Example:

$$A = aI : L_2([-\pi,\pi]) \to L_2([-\pi,\pi])\,,$$

where

$$a(x) = \begin{cases} -\frac{3}{2} + \frac{1}{2}\cos\sqrt{5}\,x\,, & -\pi \le x < y\,, \\ 2 + \cos\sqrt{2}\,x\,, & y \le x < \pi\,, \end{cases}$$

for some $y \in [-\pi/2, 0]$.

Levitin's question

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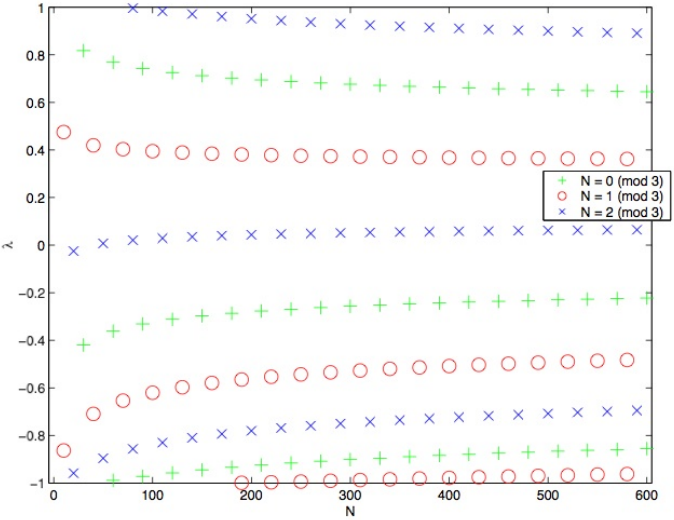
It is clear that

$$\mathrm{Spec}(A) = \mathrm{Spec}_{\mathrm{ess}}(A) = a\,([-\pi, \pi]) = [-2, -1] \cup [1, 3] \,.$$

Levitin's question



$y = -\pi/3 \quad \Longrightarrow \quad$ "3–periodicity" in N

Levitin's question

$y = -\pi q/p \quad \Longrightarrow \quad$ "periodicity" in N with the "period"

$$\omega(q/p) := \begin{cases} 2, & \text{if} \quad q = 0, \\ p, & \text{if} \quad p \text{ and } q \text{ are both odd,} \\ 2p, & \text{if} \quad q \neq 0 \text{ and either } p \text{ or } q \text{ is even.} \end{cases}$$

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$\omega(q/p)$ is the smallest natural number such that

$$\omega \times (\text{length of } [-\pi, -\pi q/p])$$
$$\text{and}$$
$$\omega \times (\text{length of } [-\pi q/p, \pi])$$

are integer multiples of 2π.

Levitin's question

Let

$$b(x) = \begin{cases} -1, & -\pi \leq x < y\,, \\ 1, & y \leq x < \pi\,. \end{cases}$$

Then

$$\widehat{b}_0 = -2y, \qquad \widehat{b}_n = \frac{2i\,(-1)^n}{n}\left(1 - e^{in(\pi - y)}\right), \quad n \in \mathbb{Z} \setminus \{0\}.$$

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Question:

Do we really have "periodicity" of eigenvalues in the gap? If yes, why?

Level sets of the resolvent norm

Let $A : X \to X$ be a bounded linear operator on a Banach space X.

J. Globevnik 1976, A. Böttcher 1994:

Can the level set

$$\{\lambda \in \mathbb{C} : \ \|(A - \lambda I)^{-1}\| = 1/\varepsilon\}$$

have an open subset?

$$\Updownarrow$$

Can $\|(A - \lambda I)^{-1}\|$ take a finite constant value on an open set?

Level sets of the resolvent norm

J. Globevnik 1976:
No, if X (or X^*) is complex uniformly convex, or if the resolvent set of A is connected $\Longrightarrow$
No, if X is a Hilbert spaces, $X = L_p(S, \Sigma, \mu)$ with $1 \le p \le \infty$, where (S, Σ, μ) is an arbitrary measure space, or if X is finite dimensional.

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ES 2008: **Yes**
ES and S. Shkarin 2009: **Yes**, even if X is separable, reflexive and strictly convex.

Level sets of the resolvent norm

Example (ES 2008) Let $B : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$,

$B = \operatorname{diag}(B_1, B_2, B_3, \dots)$ with $B_n = \begin{pmatrix} 0 & \alpha_n \\ \beta_n & 0 \end{pmatrix}$,

where $2 \le \alpha_n \to \infty$ as $n \to \infty$ and $\beta_n := 1 + 1/\alpha_n$.

Then B is a closed operator with the dense domain

$$\operatorname{Dom}(B) = \left\{ x = (x_1, x_2, \dots) \in \ell_2(\mathbb{N}) \middle| \sum_{n=1}^{\infty} \alpha_n^2 |x_{2n}|^2 < \infty \right\}$$

and

$$\boxed{\|(B - \lambda I)^{-1}\| = 1, \quad |\lambda| < \tfrac{1}{2}\,.}$$

Level sets of the resolvent norm

Question (J. R. Partington):

Let $\mathcal{H}$ be a Hilbert space and suppose a closed densely defined operator $A : \mathcal{H} \to \mathcal{H}$ is a generator of a semigroup. Can $\|(A - \lambda I)^{-1}\|$ be constant on an open set?

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Hille-Yosida: $\|(A - \lambda I)^{-1}\| = O(1/\lambda)$ as $\lambda \to +\infty$.

Level sets of the resolvent norm

Theorem.

Let $\Omega \subset \Omega_0 \subset \mathbb{C}$ be open sets and Ω_0 be connected. Let Z be a Banach space. Suppose $F : \Omega_0 \to Z$ is an analytic vector valued function,

$$\|F(\lambda)\| \leq M, \quad \forall \lambda \in \Omega$$

and $\|F(\mu)\| < M$ for some $\mu \in \Omega_0$. Then

$$\|F(\lambda)\| < M, \quad \forall \lambda \in \Omega.$$

Perturbations of the derivative of the periodic Hilbert transform: negative eigenvalues

Let

$$A\left(\sum_{k=-\infty}^{\infty} c_k e^{ikt}\right) := \sum_{k=-\infty}^{\infty} |k| c_k e^{ikt}$$

$\sim$ first order ΨDO on the unit circle $\mathbb{T}$ with the symbol $|\xi|$;

$\sim \sqrt{-\triangle}$ on $\mathbb{T}$.

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Question:

Is there a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(\tau) \to +\infty$ as $\tau \to +\infty$ and

$$\boxed{\text{Number of negative eigenvalues of } A - qI \geq h\left(\|q\|_{L^1(\mathbb{T})}\right),}$$

$\forall q \geq 0,\ \ q \in C^\infty(\mathbb{T})$?


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DE
CHATEAU LATOUR
PAUILLAC
1996
1881


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DE
CHATEAU LATOUR
PREMIER GRAND CRU CLASSÉ
APPELLATION PAUILLAC CONTROLEE
PAUILLAC-MÉDOC
1958
MIS EN BOUTEILLES AU CHATEAU
SOCIÉTÉ CIVILE DU VIGNOBLE DE CHÂTEAU LATOUR
PROPRIÉTAIRE A PAUILLAC-GIRONDE