Solving Rational Eigenvalue Problems via Linearization

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Outline

- SEP: $(A - \lambda I)x = 0$
- GEP: $(A - \lambda B)x = 0$
- QEP: $(\lambda^2 M + \lambda D + K)x = 0$
- PEP:

$$\left(\sum_{i=0}^{d} \lambda^i A_i \right) x = 0$$

- REP:

$$\left(A - \lambda B - \sum_{i=1}^{k} \frac{\lambda}{\lambda - \sigma_i} C_i \right) x = 0$$

- NEP:

$$T(\lambda)x = 0$$

QEP:

$$\left(\lambda^2 M + \lambda D + K\right) x = 0$$

Linearization: first canonical form

$$\left(\lambda \begin{bmatrix} M & \\ & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & \end{bmatrix}\right) \begin{bmatrix} \lambda x \\ x \end{bmatrix} = 0$$

- Many linearizations
- MATLAB function `polyeig` uses the 1st canonical form and then call `eig`, by Higham & Tisseur
- a very active research topic

Picard iteration: given λ_0,

$$(A - \lambda_{k+1}B - s(\lambda_k)E)\,x = 0, \quad k = 0, 1, 2, \ldots$$

Newton method:

$$\begin{cases} T(\lambda_k)x_{k+1} + (\lambda_{k+1} - \lambda_k)T'(\lambda_k)x_k = 0 \\ c^T x_{k+1} - 1 = 0 \end{cases}$$

Successive linearization approximation method (SLAM):

$$\begin{cases} \left[T(\lambda_k) + (\lambda_{k+1} - \lambda_k)T'(\lambda_k)\right] x_{k+1} = 0 \\ c^T x_{k+1} - 1 = 0 \end{cases}$$

General algorithms

- Nonlinear inverse iteration
- Nonlinear Rayleigh Quotient iteration
- Nonlinear Arnoldi method
- Nonlinear Jacobi-Davidson algorithm
- Nonlinear conjugate gradient method, $\cdots$

Problems:

- global convergence?
- convergent to a desired eigenvalue?
- all desired are computed? any lost?
- structure preservation and utilization?

$$R(\lambda)x = \left(P(\lambda) - \sum_{i=1}^{k} \frac{s_i(\lambda)}{q_i(\lambda)} E_i \right) x = 0$$

- Brute-force way: Multiply with $\prod_{i=1}^{k} q_i(\lambda)$ to get a PEP
  - very high degree if k is large, e.g. $k = 9$
  - all matrices mixed
  - structures?
- Looking REPs as NEPs
  - nonlinear methods
  - Convergence? Any lost?

Linearization of REP

Via linearization,

- equivalence: no eigenvalue lost
- easy computation: standard algorithms and software, e.g. ARPACK

New linearization vs brute-force way + PEP linearization

- smaller-size LEP
- structure preservation
  $\rightsquigarrow$ properties preservation
  $\rightsquigarrow$ better algorithms
  $\rightsquigarrow$ faster convergence

Case I: Loaded Elastic String

REP from loaded elastic string:

$$\left(A - \lambda B + \frac{\lambda}{\lambda - 1} e_n e_n^T\right) x = 0,$$

- A, B: SPD, tridiagonal
- e_n: the last column of identity

S. I. Solov'ëv. Preconditioned iterative methods for a class of nonlinear eigenvalue problems. *Linear Algebra Appl.*, 415:210-229, 2006

T. Betcke and N. J. Higham and V. Mehrmann and C. Schroder and F. Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. `http://www.manchester.ac.uk/mims/eprints`, 2008.

REP:

$$\left(A - \lambda B + \frac{\lambda}{\lambda - 1} e_n e_n^T\right) x = 0,$$

Rewrite in proper form

$$(A + e_n e_n)x - \lambda Bx + e_n \underline{e_n^T x/(\lambda - 1)} = 0$$

New variable:

$$y = e_n^T x/(\lambda - 1) \quad \Leftrightarrow \quad e_n^T x + y - \lambda y = 0$$

Linearization of REP:

$$(\mathcal{A} - \lambda \mathcal{B})z = \left(\begin{bmatrix} A + e_n e_n^T & e_n \\ e_n^T & 1 \end{bmatrix} - \lambda \begin{bmatrix} B & \\ & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

- Structure preservation: $\mathcal{A}$ and $\mathcal{B}$ are SPD and tridiagonal

10 computed smallest eigenvalues by MATLAB function `eig` for $n = 100$:

i	$\widehat{\lambda}_i$	residual
1	0.457318488953671	$5.58e-013$
2	4.48217654587198	$5.96e-013$
3	24.2235731125539	$6.69e-013$
4	63.7238211419405	$9.40e-013$
5	123.031221067605	$8.63e-013$
6	202.200899143561	$9.56e-013$
7	301.310162794155	$1.09e-012$
8	420.456563106511	$1.01e-012$
9	559.757586307048	$7.12e-013$
10	719.350660116386	$9.15e-013$

Full precision!

CPU time

	solver	$n = 200$	$n = 400$	$n = 600$	$n = 800$
$\mathcal{A} - \lambda\mathcal{B}$	`eig`	0.0156	0.1406	1.0625	3.5469
QEP	`polyeig`	0.7500	6.8594	24.5313	84.9063
$A - \lambda B$	`eig`	0.0156	0.1406	1.0469	3.5313

with new linearization, solving this REP is as easy as solving an LEP !

Rational Eigenvalue Problem

$$\left(A - \lambda B - \sum_{i=1}^{k} \frac{\lambda}{\sigma_i - \lambda} C_i C_i^T \right) x = 0$$

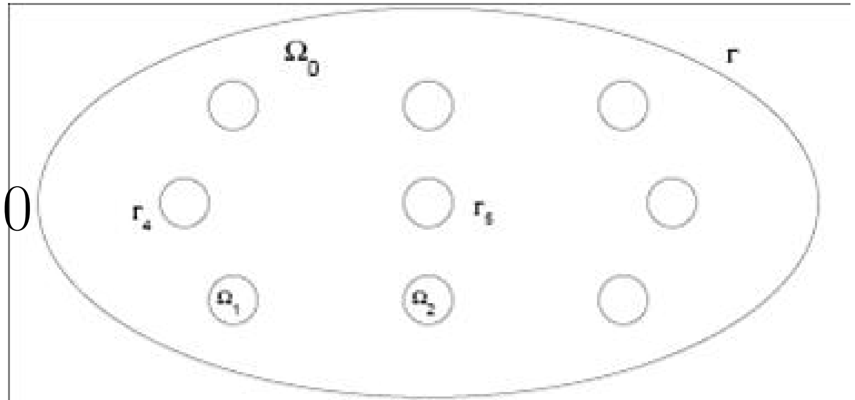


Fig. 1: Domain Ω_0

- A, B: SPD; C_i: low rank; $\sigma_i > 0$

An example

- number of rational terms $k = 9$; poles $\sigma_i = i, i = 1 : 9$;
- C_i: $n \times 2$
- size $n = 36,046$;
- interested eigenvalues: all ones in $(\sigma_1, \sigma_2) = (1, 2)$.

H. Voss. A rational spectral problem in fluid-solid vibration. *E. Trans. Numer. Anal.* vol 16, pp 94-106, 2003.

REP:

$$\left(A - \lambda B + \sum_{i=1}^{k} \frac{\lambda}{\lambda - \sigma_i} C_i C_i^T\right) x = 0,$$

Rewrite in proper form:

$$\left(A + \sum_{i=1}^{k} C_i C_i^T - \lambda B + \sum_{i=1}^{k} \frac{\sigma_i}{\lambda - \sigma_i} C_i C_i^T\right) x = 0,$$

New variables:

$$y_i = \sigma_i C_i^T x / (\lambda - \sigma_i) \quad \Leftrightarrow \quad C_i^T x + y_i - \lambda y_i / \sigma_i = 0$$

Linearization of REP:

$$(\mathcal{A} - \lambda \mathcal{B}) z = \left(\begin{bmatrix} \widetilde{A} & C_1 & \cdots & C_k \\ C_1 & I_2 & & \\ \vdots & & \ddots & \\ C_k & & & I_2 \end{bmatrix} - \lambda \begin{bmatrix} B & & & \\ & I_2/\sigma_1 & & \\ & & \ddots & \\ & & & I_2/\sigma_k \end{bmatrix} \right) \begin{bmatrix} x \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = 0.$$

After new linearization

- Size after new linearization $36,046 + 2 \times 9 = 36064$;
- Structure preserved: $\mathcal{A}$ is symmetric, $\mathcal{B}$ is positive definite

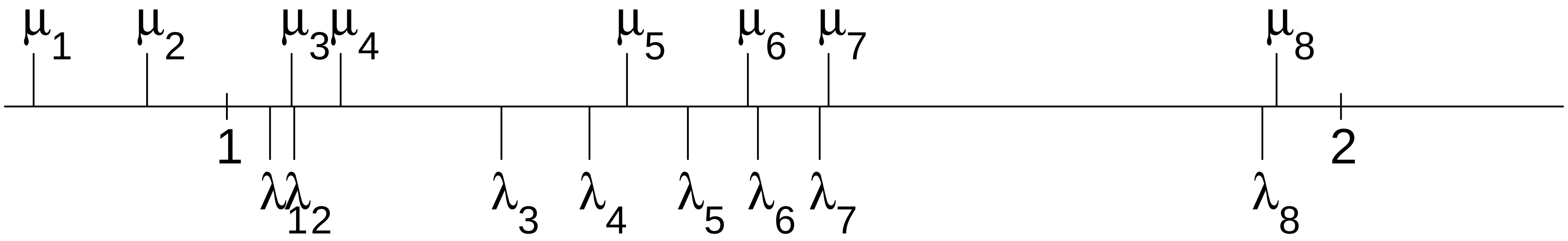
? $\mathcal{A}$ may be not sparse, however, its is a sparse matrix with low rank updating!

CPU time

	$\mathcal{A}-\lambda\mathcal{B}$	$A-\lambda B$
`ldlsparse`	0.64	0.64
preprocessing	0.49	0
`eigs`	7.14	6.66
Total	8.27	7.30

With new linearization, solving this REP is as easy as solving an LEP!

Case II



- $\lambda_1, \ldots, \lambda_8$: Linearized REP $\mathcal{A} - \lambda\mathcal{B}$;
- $\mu_1, \ldots, \mu_8$: $A - \lambda B$: linear part without rational terms.

Case III: QEP

QEP:

$$(\lambda^2 M + \lambda D + K)x = 0$$

Let $K = LR$: LU, QR, Cholesky, or Rank-revealing Decomp, $\cdots$

$$\lambda Mx + Dx + L\underline{(Rx)/\lambda} = 0$$

New variable $y = Rx/\lambda$. "Linearization" of QEP:

$$\left(\lambda \begin{bmatrix} M & \\ & -I \end{bmatrix} + \begin{bmatrix} D & L \\ R & \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

- 1st canonical form: $L = K, R = I$;
- 2nd canonical form: $L = I, R = K$;
- If L, R are of low ranks, the size is smaller than $2n$. More importantly, those explicit zero eigenvalues are kicked away!

REP:

$$R(\lambda)x = \left(P(\lambda) - \sum_{i=1}^{k} \frac{s_i(\lambda)}{q_i(\lambda)} E_i \right) x = 0.$$

- more complex linearization, involving minimal realization of rational functions, etc
- more applications
- more details
- Key: E_i is of low rank!

Yangfeng Su and Zhaojun Bai. Solving Rational Eigenvalue Problems via Linearization. http://www.cs.ucdavis.edu/research/tech-reports/2008/CSE-2008-13.pdf Submitted to SIMAX

Thank you for your attention!