Calculating Electromagnetic Radiation Fields and their Effects on Matter

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The Maxwell Equations

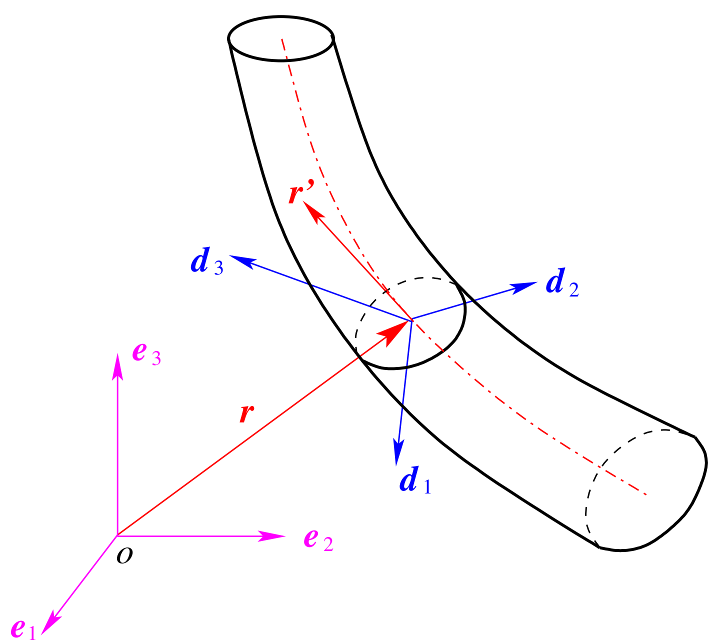
$$\boldsymbol{d} \underset{(1)}{\mathbf{e}} = -\mu \underset{(2)}{\dot{\mathbf{H}}},$$

$$\boldsymbol{d} \underset{(2)}{\mathbf{H}} = 0,$$

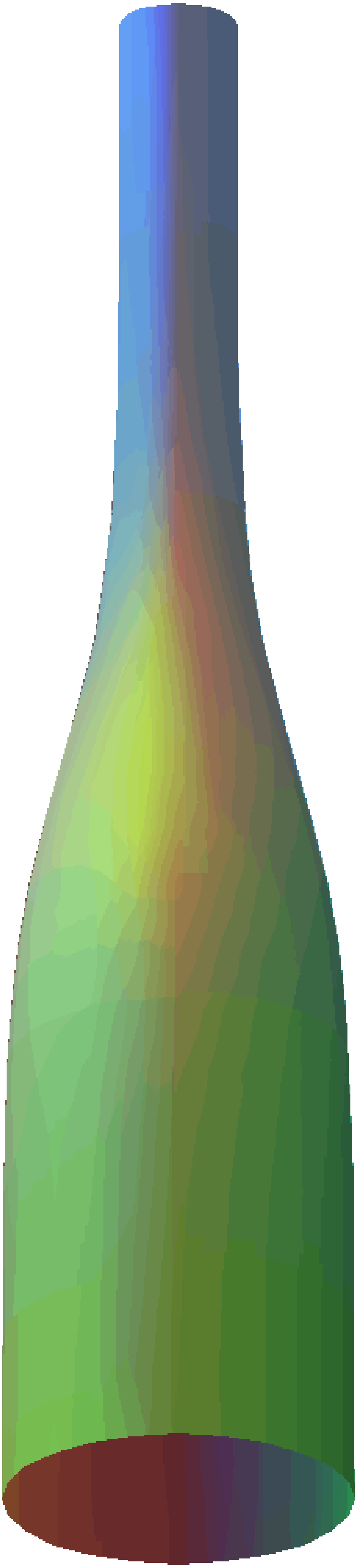
$$\boldsymbol{d} \underset{(1)}{\mathbf{h}} = \varepsilon \underset{(2)}{\dot{\mathbf{E}}} + \underset{(2)}{J},$$

$$\varepsilon \boldsymbol{d} \underset{(2)}{\mathbf{E}} = \underset{(0)}{\rho} \# 1.$$

Cavity Geometry



The dotted red space curve can be taken as a line of centroids of cross-sections spanned by the unit vectors d_1 and d_2 belonging to a Frenet triad along this curve.



Ultra-relativistic Charged Distributions in Tapering Geometries

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Introduction

Efficient design of accelerator components such as collimators depends on understanding the consequences of passing anultra-relativisticc charged beam through a tapered waveguide. This is currently studied using a combination of experiment and computer simulation. However, analytical treatments have also been explored (see, for example, [1]). Most recently, Stupakov [2] developed a process for evaluating the impedance up to the second order of iteration for low frequency beams travelling at $v = c$ in perfectly conducting waveguides of arbitrary cross-section. Using a similar method, we develop a scheme for obtaining the electromagnetic field (and hence wake potential or impedance) of a gradually tapered, axisymmetric geometry containing a bunch of arbitrary profile travelling at the speed of light parallel to the axis of the taper. Coordinate-free expressions for Maxwell's equations are 2+2-split in a coordinate system adapted to the particle beam. Our choice of auxiliary potentials ensures that an asymptotic expansion for a gradual taper yields a coupled hierarchy of Poisson equations. This can then be solved to arbitrary order of iteration, and does not require a low frequency condition.

Field Equations and Boundary Conditions

The co-ordinate free Maxwell equations for the space-time 2-form F are

$$\mathrm{d}F = 0, \quad \mathrm{d}\star F = -\frac{\rho}{\varepsilon_0}\star\tilde{V} \tag{1}$$

where ε_0 is the permittivity of free space, and the the source has the charge density ρ and 4-velocity 1-form $\tilde{V}$. $\star$ is the Hodge map on space-time and d is the exterior derivative. The source and confining geometry are most conveniently expressed in the co-ordinate chart (r,θ,ζ,u). In Cartesians, $\zeta \equiv z$, $u \equiv z - ct$. We choose units where $c = 1$. The Minkowski metric is and volume 4-form are given in this chart by

$$g = \mathrm{d}\zeta\otimes\mathrm{d}u + \mathrm{d}u\otimes\mathrm{d}\zeta - \mathrm{d}u\otimes\mathrm{d}u + g_\perp, \quad \star 1 = \mathrm{d}\zeta\wedge\mathrm{d}u\wedge\#_\perp 1 \tag{2}$$

where $g_\perp$ and $\#_\perp$ are the induced metric and Hodge map on the transverse, cross-sectional domain $\mathcal{D}$ at fixed u and ζ, with

$$g_\perp = \mathrm{d}r\otimes\mathrm{d}r + r^2\mathrm{d}\theta\otimes\mathrm{d}\theta, \quad \#_\perp 1 = r\mathrm{d}r\wedge\mathrm{d}\theta \tag{3}$$

An ultra-relativistic source moving along the ζ axis can be modelled by taking

$$\tilde{V} = \mathrm{d}u \tag{4}$$

As $\star\mathrm{d}u = \mathrm{d}u\wedge\#_\perp 1$, the second equations of (1) implies that $\partial_\zeta\rho = 0$ so that the charge density can be written $\rho(r,\theta,u)$ and is invariant under translation in the ζ-direction. F can be uniquely decomposed

$$F = \Phi\mathrm{d}\zeta\wedge\mathrm{d}u + \mathrm{d}u\wedge(\mathrm{d}_\perp A + \#_\perp\mathrm{d}_\perp a) + \mathrm{d}\zeta\wedge(\mathrm{d}_\perp B + \#_\perp\mathrm{d}_\perp b) + \Psi\#_\perp 1 \tag{5}$$

for 0-forms A, a, B, b, Φ and Ψ provided A and B vanish on the boundary $\partial\mathcal{D}$ [4]. This condition is compatible with the perfectly conducting boundary conditions that will be imposed on F below.
Without loss of generality, it proves expedient to re-write the form of F in terms of six new fields W, X, $\mathcal{H}^B$, $\mathcal{H}^b$, $\mathcal{H}^\Phi$ and $\mathcal{H}^\varphi$ that will facilitate our subsequent analysis;

$$\begin{gathered} A = \partial_u W + \partial_\zeta W - \mathcal{H}^B, \quad B = \mathcal{H}^B - \partial_\zeta W \\ \Phi = \partial_u\mathcal{H}^\Phi + \partial_u\mathcal{H}^B + \partial_\zeta\mathcal{H}^B - 2\partial^2_{u\zeta}W - \partial^2_{\zeta\zeta}W \\ a = \partial_u X, \quad b = \partial_\zeta X - \mathcal{H}^\varphi \\ \Psi = \partial_\zeta\mathcal{H}^\varphi + \partial_u\mathcal{H}^\varphi + \mathcal{H}^b - 2\partial^2_{u\zeta}X - \partial^2_{\zeta\zeta}X \end{gathered} \tag{6}$$

Thus, the Maxwell equations then reduce to the following relations:

$$\begin{gathered} \delta_\perp\mathrm{d}_\perp\mathcal{H}^B = 0, \quad \mathrm{d}_\perp\mathcal{H}^b = \#_\perp\mathrm{d}_\perp\left(\partial_u\mathcal{H}^B\right), \quad \mathrm{d}_\perp\mathcal{H}^\varphi = \#_\perp\mathrm{d}_\perp\mathcal{H}^\Phi \\ \delta_\perp\mathrm{d}_\perp W - 2\partial^2_{u\zeta}W - \partial^2_{\zeta\zeta}W + \partial_u\mathcal{H}^\Phi + \partial_u\mathcal{H}^B + \partial_\zeta\mathcal{H}^B = P(r,\theta,u) \\ \delta_\perp\mathrm{d}_\perp X - 2\partial^2_{u\zeta}X - \partial^2_{\zeta\zeta}X + \partial_\zeta\mathcal{H}^\varphi + \partial_u\mathcal{H}^\varphi + \mathcal{H}^b = 0 \end{gathered} \tag{7}$$

where $\partial_u P(r,\theta,u) = \frac{\rho(r,\theta,u)}{\varepsilon_0}$. The second equation in (7) implies the harmonic equations

$$\delta_\perp\mathrm{d}_\perp\mathcal{H}^b = \delta_\perp\mathrm{d}_\perp\mathcal{H}^\varphi = \delta_\perp\mathrm{d}_\perp\mathcal{H}^\Phi = 0 \tag{8}$$

The waveguide wall is the space-like hypersurface $f := r - R(\zeta) = 0$ for some smooth function $R(\zeta)$. We assume a perfectly conducting boundary condition for F:

$$\mathrm{d}f\wedge F = 0 \quad \text{at} \quad f = 0 \tag{9}$$

Equation (9) can be satisfied by setting

$$W = \partial_r X = 0, \quad \mathcal{H}^B = \partial_\zeta W, \quad \mathcal{H}^\Phi = -R'(\zeta)\frac{1}{r}\partial_\theta X \tag{10}$$

on the boundary $f = 0$.

Asymptotic Expansion for a Gradual Taper

In a regular, infinite, perfectly conducting cylinder, the source and confining geometry, and hence F, are independent of ζ. If the boundary varies gradually with ζ, so that for small parameter ϵ

$$f := r - \check{R}(\epsilon\zeta) = 0 \tag{11}$$

then the electromagnetic 2-form F will also have a gradual ζ-variation. Introduce a 'slow' longitudinal co-ordinate

$$s = \epsilon\zeta \tag{12}$$

and rewrite all the potentials in terms of s, using the notation

$$\chi(r,\theta,\zeta,u) = \check{\chi}(r,\theta,s,u) \tag{13}$$

where $\check{\chi} \in \left\{\check{W}, \check{X}, \check{\mathcal{H}}^B, \check{\mathcal{H}}^b, \check{\mathcal{H}}^\Phi, \check{\mathcal{H}}^\varphi\right\}$. Express the potentials in the form of asymptotic series in ϵ:

$$\check{\chi} = \sum_{n=0}^{\infty}\epsilon^n\check{\chi}_n \tag{14}$$

Note $\partial_\zeta\chi = \epsilon\check{\chi}'$ (where, from now on, a prime denotes differentiation with respect to s). The Maxwell equations (7) with boundary conditions (10) decouple to yield a hierarchical set of 2-dimensional Laplace and Poisson equations for every order n, and the boundary conditions on $\check{\mathcal{H}}^B_n$ and $\check{\mathcal{H}}^\Phi_n$ depend on $(n-1)$-order potentials. This leads to a straightforward procedure for calculating the potentials order-by-order. For $n = 0$, the only non-zero potential is $\check{W}_0$ which is a solution to $\delta_\perp\mathrm{d}_\perp W_0 = P(r,\theta,u)$ and vanishes at $r = R(s)$.
For every subsequent order of n:

1. Calculate the harmonic potential $\check{\mathcal{H}}^B_n$ by solving the 2-dimensional Laplace equation
$$\delta_\perp\mathrm{d}_\perp\check{\mathcal{H}}^B_n = 0 \tag{15}$$
subject to the boundary condition
$$\check{\mathcal{H}}^B_n = \check{W}'_{n-1} \quad \text{at } r = \check{R}(s) \tag{16}$$
2. Calculate $\check{\mathcal{H}}^b_n$ from
$$\mathrm{d}_\perp\check{\mathcal{H}}^b_n = \partial_u\#_\perp\mathrm{d}_\perp\check{\mathcal{H}}^B_n \tag{17}$$
3. Calculate the harmonic potential $\check{\mathcal{H}}^\Phi_n$ by solving the 2-dimensional Laplace equation
$$\delta_\perp\mathrm{d}_\perp\check{\mathcal{H}}^\Phi_n = 0 \tag{18}$$
subject to $\check{\mathcal{H}}^\Phi_n = -\check{R}'(s)\frac{1}{r}\partial_\theta\check{X}_{n-1}$ at $r = \check{R}(s)$
4. Calculate $\check{\mathcal{H}}^\varphi_n$ from
$$\mathrm{d}_\perp\check{\mathcal{H}}^\varphi_n = \#_\perp\mathrm{d}_\perp\check{\mathcal{H}}^\Phi_n \tag{19}$$
5. Calculate the potential W_n by solving the 2-dimensional Poisson equation
$$\delta_\perp\mathrm{d}_\perp\check{W}_n = \check{W}''_{n-2} + 2\partial_u\check{W}'_{n-1} - \partial_u\check{\mathcal{H}}^\Phi_n - \partial_u\check{\mathcal{H}}^B_n - \check{\mathcal{H}}^{B\prime}_{n-1} \tag{20}$$
where $\check{W}_n$ vanishes at $r = \check{R}(s)$.
6. Calculate the potential $\check{X}_n$ by solving the 2-dimensional Poisson equation
$$\delta_\perp\mathrm{d}_\perp\check{X}_n = \check{X}''_{n-2} + 2\partial_u\check{X}'_{n-1} - \check{\mathcal{H}}^b_n - \partial_u\check{\mathcal{H}}^\varphi_n - \check{\mathcal{H}}^{\varphi\prime}_{n-1} \tag{21}$$
with $\partial_r\check{X}_n = 0$ at $r = \check{R}(s)$

References

[1] Stupakov G V 1995 *Geometrical Wake of a Smooth Taper* SLAC-PUB-95-7086

[2] Stupakov G V, Phys. Rev. ST Accel. Beams **8** 094401 (2007)

[3] Tucker R W, Theoretical and Applied Mechanics, 34 (1):1-49 (2007)

[4] Abraham R, Marsden J E, Ratiu T 1988 *Manifolds, Tensor Analysis and Applications* (New York: Springer-Verlag)

[5] http://abci.kek.jp/abci.htm

Example

The fields (and hence wakes and impedances) can be evaluated to arbitrary order in the gradual-taper asymptotic series. In Proceedings article, impedances are calculated for a harmonic line current offset from the central axis of an axisymmetric, perfectly conducting waveguide of arbitrary, smooth, shallow taper. The leading order terms exactly match the results in [2]. Higher order corrections then obtained.

To further test the scheme, the logitudinal wake potential was calculated for a Gaussian beam travelling along the central axis of a waveguide with a taper given by

$$R(\zeta) = 20 - 18\ \mathrm{sech}\ (0.01\zeta) \tag{22}$$

where the distances are expressed in millimetres.

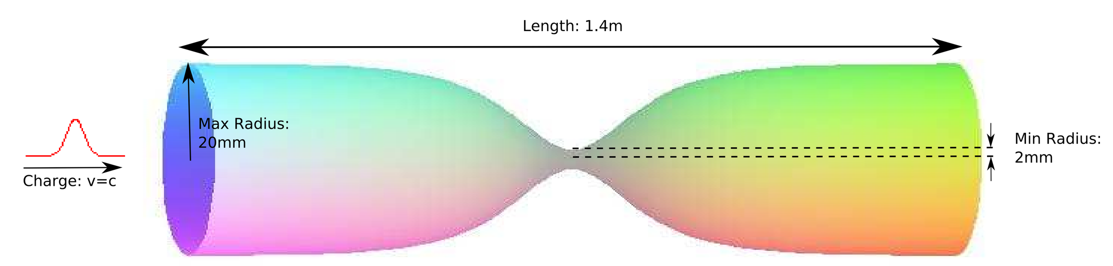


Figure 1: A Collimator with a "sech" Taper

The wakefields were calculated using both the asymptotic expansion scheme and the Azimuthal Beam Cavity Interaction (ABCI) package [5]. The asymptotic expansion results are shown in Figure 2. The red line shows the leading order term, which is the same as the total monopole result of [2]. The green line shows the result after summing the first six terms of the asymptotic series. A typical ABCI result is shown in Figure 3. Our leading order term reaches a maximum/minimum of ± 1.451V/pC at ± 1.8mm from the bunch centre. Our sixth order result shows a maximum/minimum of ± 1.598V/pC at ± 1.7mm from the bunch centre. The plots of longitudinal wakefield obtained from ABCI vary in scale according to the choice of resolution. By performing the simulation at various resolutions, it is possible to estimate the results as the cell size tends to zero. Such a convergence study gives maximum of 1.635V/pC and a minimum of -1.615V/pC, at ± 1.7mm from the bunch source. This would suggest a good agreement between the sixth order results and the ABCI simulation.

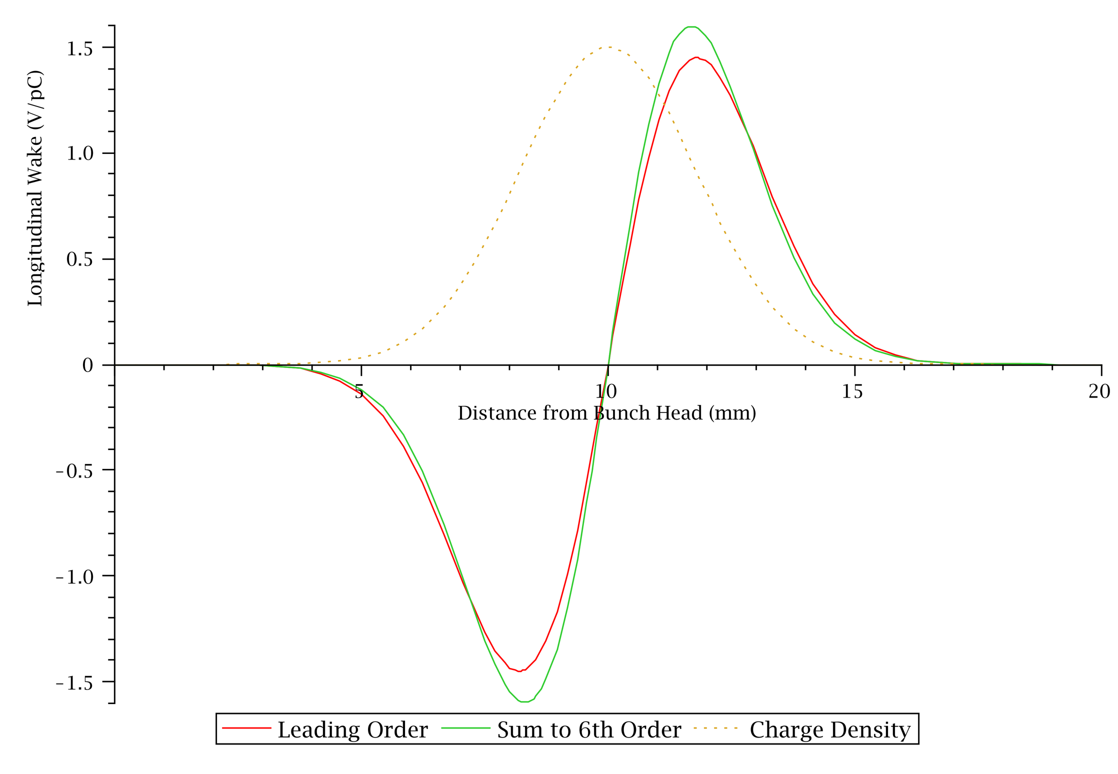


Figure 2: Longitudinal Wake Potential for a Gaussian Beam in a "sech" taper

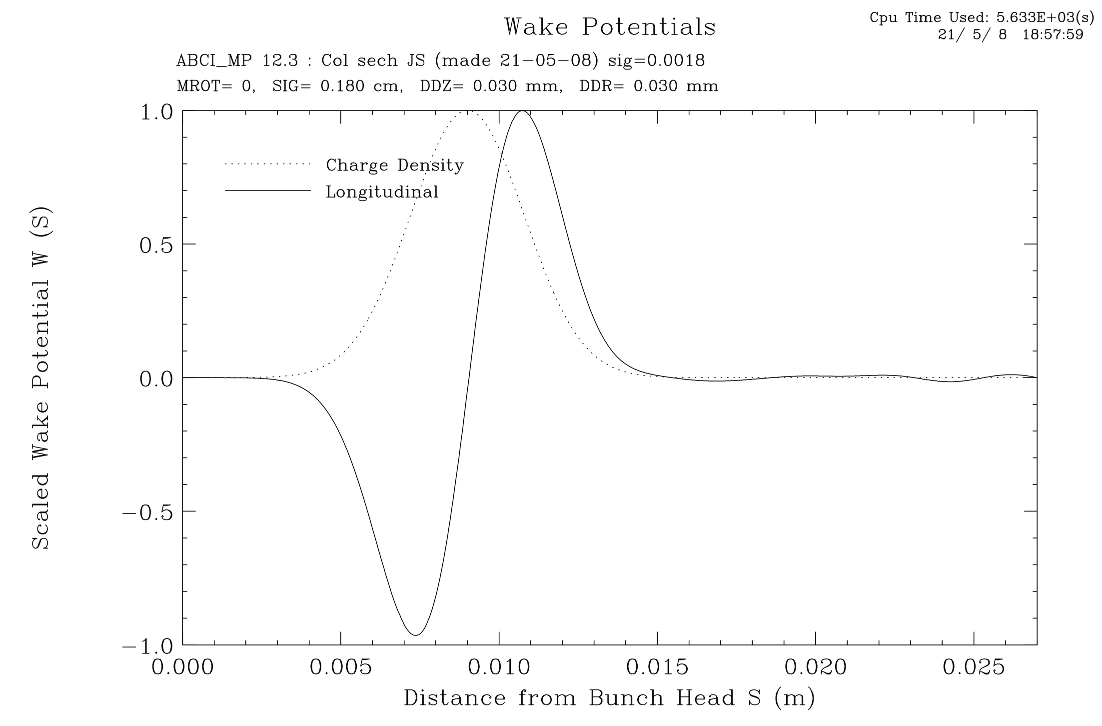


Figure 3: ABCI Longitudinal Wake Potential

- Choose a coordinate system adapted to the interior $\mathcal{U}$ of a beam pipe with a circular disc cross-section of fixed radius a at every point and an axis given by a planar space-curve with, in general, non-constant curvature κ and $|\kappa a| \ll 1$.
- At each point on this curve one may erect a triad of orthogonal vectors in space, one member of which is tangent to the curve. The remaining vectors define a transverse plane.
- All points in the interior $\mathcal{U}$ of the beam pipe lie on some transverse plane associated with such a triad with origin at some point on the axial space-curve.
- Let the region $\mathcal{U} \subset \mathbb{R}^3$ inside the beam pipe be described in terms of coordinates (r, θ, z) adapted to the central space-curve with curvature $\kappa(z)$ such that

$$0 \leq r \leq a, \qquad 0 < \theta \leq 2\pi, \qquad -\infty \leq z \leq \infty.$$

- A convenient field of orthonormal coframes on $\mathcal{U}$ is given in these coordinates by

$$\{e^1 = \boldsymbol{d}\, r, \quad e^2 = r\boldsymbol{d}\,\theta, \quad e^3 = (1 - \epsilon\kappa_0(z)x_1)\,\boldsymbol{d}\,z\},$$

  with $x_1 = r\cos\theta$.
- Thus the Euclidean metric tensor $\underline{g}$ on $\mathcal{U}$ is given by

$$\underline{g} = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3.$$

- In these coordinates the pipe boundary is the surface $r = a$, the coordinate z measures arc-length along the space-curve and on the space-curve $r = 0$.

- The objective is to solve Maxwell's equations for the fields $\underset{(1)}{\mathbf{e}}$ and $\underset{(1)}{\mathbf{h}}$ on $\mathcal{U}$ in terms of prescribed sources and initial data as a perturbative expansion in the axial curvature of the beam pipe.
- The strategy will be to project the field system into suitable modes that ensure that perfectly conducting boundary conditions are satisfied at the pipe boundary.
- In the adapted coordinate system this is achieved with the aid of complex Dirichelet and Neumann eigen-modes of the two-dimensional Laplacian associated with each transverse plane in the beam pipe.

Dirichelet Modes

- A *complex Dirichelet mode set* $\{\Phi_N\}$ is a collection of complex eigen 0-forms of the Laplacian operator on the disc $\mathcal{D}$ that vanishes on the boundary $\partial\mathcal{D}$:

$$\Delta\Phi_N + \beta_N^2\Phi_N = 0$$

  with $\Phi_N|_{\partial\mathcal{D}} = 0$.
- This boundary condition and the nature of the domain determine the associated (positive non-zero real) eigenvalues β_N^2. The label N here consists of an ordered pair of real numbers.

Dirichelet Modes

- An explicit form for Φ_N is for $n \in \mathbb{Z}$

$$\Phi_N(r, \theta) = J_n \left(x_{q(n)} \frac{r}{a} \right) e^{in\theta}, \tag{1}$$

  where $J_n(x)$ is the n-th Bessel function
- the numbers $\{x_{q(n)}\}$ are defined by $J_n(x_{q(n)}) = 0$
- $N := \{n, q(n)\}$.
- The eigenvalues are given by $\{\beta_N = x_{q(n)}/a\}$.

Neumann Modes

- A *Neumann mode set* $\{\Psi_N\}$ is a collection of eigen 0-forms of the Laplacian operator on $\mathcal{D}$ such that $\frac{\partial \Psi_N}{\partial n}$ vanishes on $\partial\mathcal{D}$:

$$\Delta\Psi_N + \alpha_N^2 \Psi_N = 0$$

- This alternative boundary condition and the nature of the domain determine the associated (positive non-zero real) eigenvalues α_N^2 where again the label N consists of an ordered pair of real numbers.

Neumann Modes

- An explicit form for Ψ_M is for $m \in \mathbb{Z}$

$$\Psi_M(r,\theta) = J_m\left(x'_{p(m)}\frac{r}{a}\right)e^{im\theta}$$

  where the numbers $\{x'_{p(m)}\}$ are defined by $J'_m(x'_{p(m)}) = 0$ and $M := \{m, p(m)\}$.
- The eigenvalues are given by $\{\alpha_M = x'_{p(m)}/a\}$

Mode Decompositions

- Since $\mathcal{U}$ is simply connected one can represent the electromagnetic forms

$$
\begin{aligned}
\underset{(1)}{\mathbf{e}}(\epsilon,t,z,r,\theta) &= \sum_N V_N^E(\epsilon,t,z)\,\boldsymbol{d}\,\Phi_N + \\
&\sum_M V_M^H(\epsilon,t,z)\#(\boldsymbol{d}\,z\wedge\boldsymbol{d}\,\Psi_M) \\
&+\sum_N \gamma_N^E(\epsilon,t,z)\Phi_N(r,\theta)\,\boldsymbol{d}\,z, \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\underset{(1)}{\mathbf{h}}(\epsilon,t,z,r,\theta) &= \sum_N I_N^E(\epsilon,t,z)\#(\boldsymbol{d}\,z\wedge\boldsymbol{d}\,\Phi_N) + \\
&\sum_M I_M^H(\epsilon,t,z)\,\boldsymbol{d}\,\Psi_M \\
&+\sum_M \gamma_M^H(\epsilon,t,z)\Psi_M(r,\theta)\,\boldsymbol{d}\,z. \qquad (3)
\end{aligned}
$$

Perturbative Expansions

- Since for small $|\kappa a|$ the beam pipe approximates a straight cylinder we adopt the perturbative field-mode expansions

$$V_N^E(\epsilon, t, z) = V_N^{E(0)}(t, z) + \epsilon V_N^{E(1)}(t, z) + \mathcal{O}(\epsilon^2). \tag{4}$$

$$I_N^E(\epsilon, t, z) = I_N^{E(0)}(t, z) + \epsilon I_N^{E(1)}(t, z) + \mathcal{O}(\epsilon^2). \tag{5}$$

$$\gamma_N^E(\epsilon, t, z) = \gamma_N^{E(0)}(t, z) + \epsilon \gamma_N^{E(1)}(t, z) + \mathcal{O}(\epsilon^2), \tag{6}$$

- with analogous expansions for the magnetic modes V_M^H, I_M^H, γ_M^H

Perturbative Expansions

- Also express the sources as a power series in ϵ:

$$
\begin{aligned}
J_\theta(\epsilon, t, z, r, \theta) &= J_\theta^{(0)}(t, z, r, \theta) + \epsilon J_\theta^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2), \\
J_r(\epsilon, t, z, r, \theta) &= J_r^{(0)}(t, z, r, \theta) + \epsilon J_r^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2), \\
J_0(\epsilon, t, z, r, \theta) &= J_0^{(0)}(t, z, r, \theta) + \epsilon J_0^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2), \\
\rho(\epsilon, t, z, r, \theta) &= \rho^{(0)}(t, z, r, \theta) + \epsilon \rho^{(1)}(t, z, r, \theta) + \mathcal{O}(\epsilon^2).
\end{aligned}
$$

Telegraph Type Equations

- To first order in ϵ the problem can now be reduced to solving initial-value problems for the decoupled fields $\gamma_N^{H(0)}, \gamma_N^{H(1)}, \gamma_N^{E(0)}$ and $\gamma_N^{E(1)}$.
- For some real constant $\sigma > 0$ each satisfies a second-order hyperbolic partial differential equation in the independent variables (t, z), of the form:

$$\ddot{f} - c^2 f'' + c^2 \sigma^2 f = g, \tag{7}$$

for some prescribed source function g.

Telegraph Type Solutions

- The causal solution of this partial differential equation for $t > 0$, is determined by with prescribed values of $f(0, z)$ and $\dot{f}(0, z)$
- If the data and sources are sufficiently smooth the general solution may be expressed in the form

$$f(t, z) = \mathcal{H}_\sigma[f^{init}](t, z) + \mathcal{I}_\sigma[g](t, z) \tag{8}$$

where

$$\begin{aligned}\mathcal{H}_\sigma[f^{init}](t, z) &:= \frac{1}{2}\Big\{ f(0, z - ct) + f(0, z + ct) \Big\} \\ &+ \frac{1}{2c}\int_{z-ct}^{z+ct} d\zeta\, \dot{f}(0, \zeta) J_0(\sigma\sqrt{c^2t^2 - (z-\zeta)^2}) \\ &- \frac{ct\sigma}{2}\int_{z-ct}^{z+ct} d\zeta\, f(0, \zeta)\frac{J_1(\sigma\sqrt{c^2t^2 - (z-\zeta)^2}\,)}{\sqrt{c^2t^2 - (z-\zeta)^2}}\end{aligned} \tag{9}$$

and

$$\mathcal{I}_\sigma[g](t, z) := \frac{1}{2c}\int_0^t dt' \int_{z-c(t-t')}^{z+c(t-t')} d\zeta\, g(t', \zeta) J_0(\sigma\sqrt{c^2(t-t')^2 - (z-\zeta)^2}) \tag{10}$$

Data

- The functions $f(0,z), \dot{f}(0,z)$ constitute the initial $t=0$ Cauchy data in this solution and determine the $\mathcal{H}_\sigma$ contribution above.
- Typically, in an accelerating device, lowest order contributions include externally applied piecewise established magnetostatic and RF fields that are together used to guide and accelerate charges along the beam tube. In the following we assume that all $\mathcal{H}_\sigma$ contributions to the field solutions arise in lowest order.

Electromagnetic Power from Smooth Sources

- In the general situation all zero and first order fields can be calculated in terms of finite range integrals involving Bessel functions.
- It is of some interest to calculate how the instantaneous electromagnetic power flux depends on the first order curvature correction to that in a straight cylinder with smooth sources.
- This is obtained by integrating the Poynting vector field over the cross-section $\mathcal{D}$ at an arbitrary point with coordinate z.
- In terms of the Poynting 2-form

$$\underset{(2)}{\mathbf{S}}(\epsilon, t, z, r, \theta) := \underset{(1)}{\mathbf{e}}(\epsilon, t, z, r, \theta) \wedge \underset{(1)}{\mathbf{h}}(\epsilon, t, z, r, \theta), \tag{11}$$

such instantaneous power $w(\epsilon, t, z)$ is obtained by integrating $\underset{(2)}{\mathbf{S}}$ over $\mathcal{D}$:

$$w(\epsilon, t, z) := \int_{\mathcal{D}} \underset{(2)}{\mathbf{S}}(\epsilon, t, z, r, \theta)$$

Moving Point Charge Source

- Suppose the motion of a point charge is maintained on a curved path parallel to the *design-orbit* with curvature $\kappa(z)$ and constant speed v. Then

$$\rho(\epsilon, t, z, x_1, x_2) = Q(\epsilon, t)\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})\delta(z - vt),$$

with

$$Q(\epsilon, t) := \frac{Q_{tot}}{1 - \epsilon\kappa_0(vt)x_{1,0}} = Q_{tot} + \epsilon\kappa_0(vt)x_{1,0}Q_{tot} + \mathcal{O}(\epsilon^2),$$

in terms of the Cartesian three-dimensional Dirac distribution with moving point support at $(x_{1,0}, x_{2,0}, vt) = (r_0 \cos\theta_0, r_0 \sin\theta_0, vt)$, determining the location of the point charge in $\mathcal{U}$ at time t.

Moving Point Charge Source

- Then

$$\begin{aligned}\rho &= \rho^{(0)} + \epsilon\rho^{(1)} + \mathcal{O}(\epsilon^2),\\ \rho^{(0)}(z-vt, x_1, x_2) &= Q_{tot}\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})\delta(z - vt),\\ \rho^{(1)}(t, z, x_1, x_2) &= \kappa_0(vt)x_{1,0}Q_{tot}\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})\delta(z - vt).\end{aligned}$$

  or in adapted coordinates

$$(\rho\hat{\#}1)(\epsilon, t, z, r, \theta) = Q(\epsilon, t)\frac{\delta(r - r_0)}{r}\delta(\theta - \theta_0)\delta(z - vt)r\,\boldsymbol{d}\,r \wedge \boldsymbol{d}\,\theta.$$

- The associated electric current components are $J_r = J_\theta = 0$ and

$$J_0^{(0)} = v\rho^{(0)} = vQ_{tot}\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})\delta(z - vt), \tag{12}$$

$$\begin{aligned}J_0^{(1)} = v\rho^{(1)} &= v\kappa_0(vt)x_{1,0}Q_{tot}\\ &\quad\times\delta(x_1 - x_{1,0})\delta(x_2 - x_{2,0})\delta(z - vt)\end{aligned} \tag{13}$$

Ultra-relativistic Longitudinal Wake Potentials

- The wakefield formalism is designed to exploit the simplifications that arise by considering the (ultra-relativistic) limit obtained from charged sources moving at the speed of light.
- The resulting electromagnetic fields give rise to various wake-potentials from which wake-impedances may be computed for ultra-relativistic charged bunches with prescribed charged distributions.
- The formalism is based on calculating the emf induced on a spectator (test) ultra-relativistic point particle moving behind a leading ultra-relativistic charged particle with the same velocity but in general on a different orbit.
- Since our computations provide the electromagnetic fields for a point particle moving with arbitrary speed on an orbit (in general) off the pipe axis (with transverse coordinates (r_0, θ_0)) one may readily calculate the general longitudinal wake potential to the same order as the fields, by having the spectator charge, with transverse coordinates (r, θ), at a fixed longitudinal separation $\widetilde{s} > 0$ behind a right moving source particle.

Ultra-relativistic Longitudinal Wake Potentials

- The definition of the ultra-relativistic longitudinal wake potential is taken as

$$\mathcal{W}_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \widetilde{s}) := -\frac{1}{Q_{tot}} \int_{-\widetilde{s}/2}^{\infty} dz\, \mathcal{E}_z^{(r_0,\theta_0)} \left(\epsilon, \frac{z+\widetilde{s}}{c}, z, r, \theta\right), \tag{14}$$

  where $\mathcal{E}_z^{(r_0,\theta_0)}(\epsilon, t, z, r, \theta)$ is the z-component of the electric field generated by the point source with speed $v = c$ and charge Q_{tot}.
- The ultra-relativistic longitudinal impedance is

$$Z_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \omega) := \frac{1}{c} \int_0^{\infty} d\widetilde{s}\, e^{i\omega\widetilde{s}/c} \mathcal{W}_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \widetilde{s}),$$

  and the projected longitudinal mode impedances are

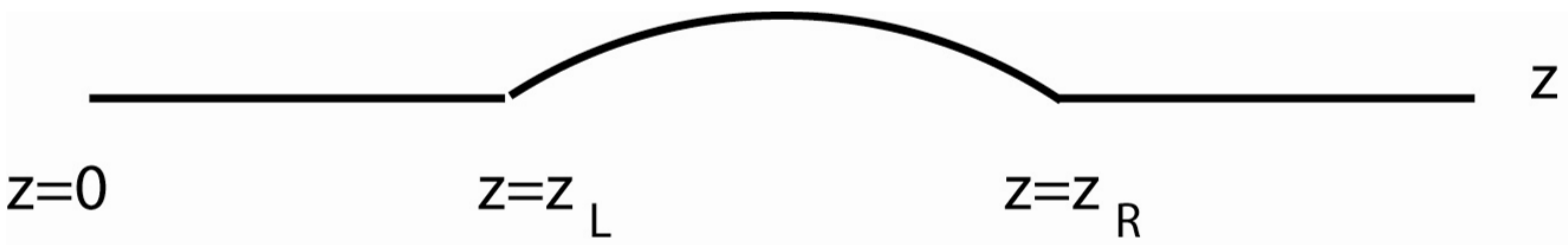
$$\langle Z_{\parallel}^{(r_0,\theta_0)} \rangle_M(\epsilon, \omega) := \int_{\mathcal{D}} Z_{\parallel}^{(r_0,\theta_0)}(\epsilon, r, \theta, \omega)\, \overline{\Phi_M}(r, \theta)\, r\, \boldsymbol{d}\, r \wedge \boldsymbol{d}\, \theta.$$

Longitudinal Wake Potential for a Pipe with Piecewise Constant Curvature

- In cases where segments of the beam pipe are connected by planar segments of arcs with constant radius of curvature one may perform these integrals analytically and hence generate *analytic expressions* for the corresponding wake impedances.
- Consider the case of an infinitely long planar pipe with axial curvature given by

$$\kappa_0(z) = \left(\Theta(z - z_L) - \Theta(z - z_R)\right) \check{\kappa}_0,$$

where $z_L, z_R, (0 < z_L < z_R)$, $\check{\kappa}_0 (\neq 0)$ are constants and $\Theta(z)$ is the Heaviside function



z
z=0
z=z L
z=z R

Longitudinal Wake Potential for a Pipe with Piecewise Constant Curvature

- With the following dimensionless variables for some length L

$$\widehat{\kappa}_0 := L\check{\kappa}_0, \quad \widehat{s} := \frac{\widetilde{s}}{L}, \quad \widehat{\beta}_M := L\beta_M, \quad \widehat{z}_R := \frac{z_R}{L}, \quad \widehat{z}_L := \frac{z_L}{L},$$

introduce the dimensionless quantities

$$\begin{aligned}
\zeta_{M,1}(\widehat{s}) \quad &:= \quad \frac{\widehat{\kappa}_0}{\widehat{\beta}_M} J_1(\sqrt{2}\widehat{\beta}_M\widehat{s}), \\
\zeta_{M,2}(\widehat{s}) \quad &:= \quad \widehat{\kappa}_0\Bigg[(\widehat{z}_R - \widehat{z}_L) - \frac{\sqrt{2}}{\widehat{\beta}_M\sqrt{\widehat{s}}} \\
&\times\Bigg\{\sqrt{\widehat{z}_R + \frac{\widehat{s}}{2}}\, J_1\left(\beta_M\sqrt{2\widehat{s}\left(\widehat{z}_R + \frac{\widehat{s}}{2}\right)}\right) \\
&+\frac{2\left(\widehat{z}_R + \frac{\widehat{s}}{2}\right)^{3/2}}{\widehat{s}} J_3\left(\widehat{\beta}_M\sqrt{2\widehat{s}\left(\widehat{z}_R + \frac{\widehat{s}}{2}\right)}\right) \\
&-\sqrt{\widehat{z}_L + \frac{\widehat{s}}{2}}\, J_1\left(\widehat{\beta}_M\sqrt{2\widehat{s}\left(\widehat{z}_L + \frac{\widehat{s}}{2}\right)}\right) \\
&-\frac{2\left(\widehat{z}_L + \frac{\widehat{s}}{2}\right)^{3/2}}{\widehat{s}} J_3\left(\widehat{\beta}_M\sqrt{2\widehat{s}\left(\widehat{z}_L + \frac{\widehat{s}}{2}\right)}\right)\Bigg\}\Bigg],
\end{aligned}$$

Longitudinal Wake Potential for a Pipe with Piecewise Constant Curvature

- Then

$$\begin{aligned}
\overline{\mathcal{W}^{(r_0,\theta_0)}_{\parallel M,edges}}(\epsilon,\widetilde{s}\,) &= \frac{\epsilon}{\sqrt{2}}\,\zeta_{M,1}(\widehat{s})\,(\breve{I}^{(r_0,\theta_0)}_M - \breve{p}^{(r_0,\theta_0)}_M - \breve{s}^{(r_0,\theta_0)}_M),\\
\overline{\mathcal{W}^{(r_0,\theta_0)}_{\parallel M,\breve{\kappa}_0}}(\epsilon,\widetilde{s}\,) &= \frac{\epsilon}{4}\zeta_{M,2}\,(\widehat{s})\,(\breve{I}^{(r_0,\theta_0)}_M - \breve{p}^{(r_0,\theta_0)}_M).
\end{aligned}$$

  Natural choices for L include $L = a$ or $L = z_R - z_L$.
- In the following Figure $\zeta_{M,1}$ and $\zeta_{M,2}$ are plotted for the choice

$$\widehat{\kappa}_0 = 1, \quad \widehat{\beta}_M = 1, \quad \widehat{z}_R = 2, \quad \widehat{z}_L = 1.$$

- In the regime $\widehat{s} \gg 1$, $\zeta_{M,2}$ tends to $\widehat{\kappa}_0(\widehat{z}_R - \widehat{z}_L)$.

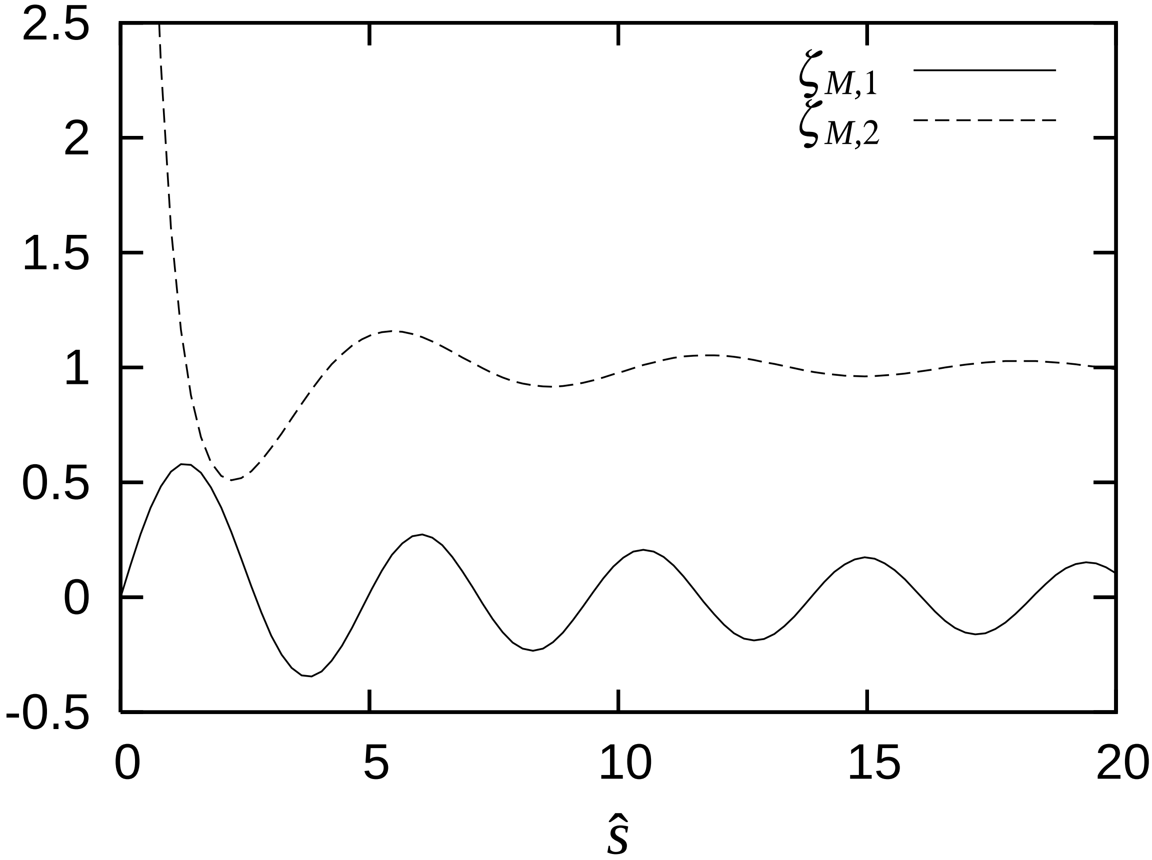


Figure 1: Dimensionless profiles for contributions to $\mathcal{W}_{\parallel\ \ M}^{(r_0,\theta_0)}(\widetilde{s})$ to $\mathcal{O}(\epsilon^2)$.

Locally Homogeneous Dispersive Magnetoelectric Media

In dispersive media, constitutive relations between the spatial fields $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{h}$ are non-local in spacetime. If the medium is *spatially homogenous*, so that it has no preferred spatial origin, then it is possible to Fourier transform the fields with respect to space and time, and work with transformed constitutive relations that are local in the space of Fourier field components.

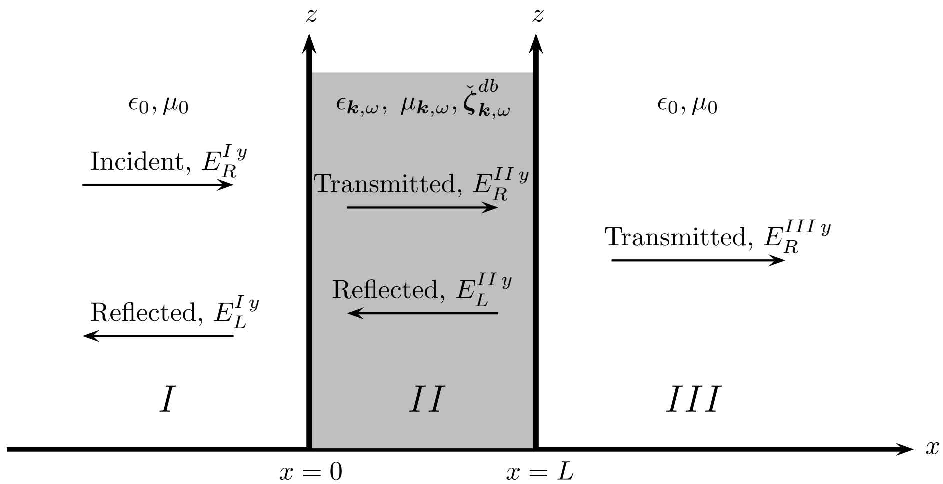


Figure 1: Geometry of the magnetoelectric slab and the electric field amplitudes in the three regions.

Locally Homogeneous Dispersion Relation for a Linear Magneto-electric Medium

- For any real valued p-form α, define its complex valued Fourier transform $\check{\alpha}_{\boldsymbol{k},\omega}$ by

$$\alpha = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\boldsymbol{k}\ \check{\alpha}_{\boldsymbol{k},\omega} \exp i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t), \tag{1}$$

  where $\boldsymbol{k} \in \boldsymbol{R}^3$.

- The source-free Maxwell system reduces to

$$\begin{aligned} \boldsymbol{K} \wedge \check{\boldsymbol{e}}_{\boldsymbol{k},\omega} &= \omega \check{\boldsymbol{B}}_{\boldsymbol{k},\omega} && (2)\\ \boldsymbol{K} \wedge \check{\boldsymbol{h}}_{\boldsymbol{k},\omega} &= -\omega \check{\boldsymbol{D}}_{\boldsymbol{k},\omega}, && (3) \end{aligned}$$

  where the real propagation wave 1-form $\boldsymbol{K} \equiv \boldsymbol{k}\cdot d\boldsymbol{r}$.

- The remaining transformed Maxwell equations $\boldsymbol{K} \wedge \check{\boldsymbol{B}}_{\boldsymbol{k},\omega} = 0$ and $\boldsymbol{K} \wedge \check{\boldsymbol{D}}_{\boldsymbol{k},\omega} = 0$ follow trivially.

Locally Homogeneous Dispersion Relation for a Linear Magneto-electric Medium

- Assume magneto-electric constitutive relations: form

$$\check{\boldsymbol{d}}_{\boldsymbol{k},\omega} = \check{\zeta}^{de}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) + \check{\zeta}^{db}_{\boldsymbol{k},\omega}(\check{\boldsymbol{b}}_{\boldsymbol{k},\omega}) \tag{4}$$

$$\check{\boldsymbol{h}}_{\boldsymbol{k},\omega} = \check{\zeta}^{he}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) + \check{\zeta}^{hb}_{\boldsymbol{k},\omega}(\check{\boldsymbol{b}}_{\boldsymbol{k},\omega}). \tag{5}$$

- By convolution, they give rise to non-local spacetime constitutive relations.

- Substituting into Maxwell's equations yields a degenerate 1-form eigen-equation for $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$:

$$\omega^2 \check{\zeta}^{de}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) + \omega \check{\zeta}^{db}_{\boldsymbol{k},\omega}\left(\#(\boldsymbol{K} \wedge \check{\boldsymbol{e}}_{\boldsymbol{k},\omega})\right) \quad + \\ + \omega \# \left(\boldsymbol{K} \wedge \check{\zeta}^{he}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega})\right) + \\ + \# \left(\boldsymbol{K} \wedge \check{\zeta}^{hb}_{\boldsymbol{k},\omega}\left(\#(\boldsymbol{K} \wedge \check{\boldsymbol{e}}_{\boldsymbol{k},\omega})\right)\right) = 0.$$

  This may be written

$$\check{\boldsymbol{\mathcal{D}}}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) = 0,$$

  defining the $1-1$ tensor $\check{\boldsymbol{\mathcal{D}}}_{\boldsymbol{k},\omega}$.
- For non-trivial solutions $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$, the determinant of the matrix $\check{\mathcal{D}}_{\boldsymbol{k},\omega}$ representing $\check{\boldsymbol{\mathcal{D}}}_{\boldsymbol{k},\omega}$ must vanish:

$$det(\check{\mathcal{D}}_{\boldsymbol{k},\omega}) = 0.$$

- In general, the roots of this dispersion relation are not invariant under the transformation $\boldsymbol{K} \to -\boldsymbol{K}$.

- If one writes $\boldsymbol{k} = \hat{\boldsymbol{k}}|\boldsymbol{k}|$ in terms of the Euclidean norm $|\boldsymbol{k}|$, and introduces the refractive index $\mathcal{N} = |\boldsymbol{k}|\frac{c_0}{\omega} > 0$ and $\hat{\boldsymbol{k}}$ in place of $\boldsymbol{k}$, then solutions propagating in the direction described by $\hat{\boldsymbol{k}}$ with angular frequency $\omega > 0$ correspond to roots that may be expressed in the form

$$\mathcal{N}_r = \mathcal{F}_r(\hat{\boldsymbol{k}}, \omega).$$

- There can be several distinct characteristic waves each with its unique refractive index that depends on the propagation direction $\hat{\boldsymbol{k}}$ and frequency ω.
- When the determinant characteristic equation is a quadratic polynomial in $\mathcal{N}^2$ and has two distinct roots that describe two distinct propagating modes for a given ω, the medium is termed *birefringent*.
- Roots $\mathcal{N}_r^2$ such that $\mathcal{N}_r(\hat{\boldsymbol{k}}, \omega) \neq \mathcal{N}_r(-\hat{\boldsymbol{k}}, \omega)$ imply that harmonic plane waves propagating in the opposite directions $\pm\hat{\boldsymbol{k}}$ have different wave speeds.
- Each eigen-wave will have a uniquely defined polarisation.

Classical Average Radiation Pressure on a magneto-electric slab

The average radiation pressure exerted by a plane polarized monochromatic plane wave that is incident normally from the left on a (suitably) oriented magneto-electric slab of finite thickness is different from that exerted by the same wave incident from the right.

Conclusions

- An *analytic perturbative* approach to the computation of electromagnetic fields generated by a variety of charged sources moving with prescribed motions in a perfectly conducting beam pipe of radius a with planar curvature $\kappa(z)$ has been presented.
- Results were given in terms of expressions involving powers of $|a\kappa(z)| \ll 1$ and $|a^2\kappa'(z)|$.
- They included a discussion of ultra-relativistic longitudinal wake potentials from which pipe impedances induced by $\kappa(z) \neq 0$ can be calculated.
- The approach been explicitly illustrated for pipes with piecewise constant curvature modeling pipes with straight segments linked by circular arcs of (arbitrary) finite length.

Conclusions

- The bi-refringent properties of linear magneto electric media were outlined and the radiation reaction of a magneto-electric slab discussed
- The investigation of many important properties of the interaction of Electromagnetic fields with matter could be facilitated by improved (non-perturbative) techniques for solving boundary-value problems for the Maxell-system.

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