Structured matrix methods for the calculation of the roots of inexact polynomials

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1. DIFFICULTIES OF COMPUTING POLYNOMIAL ROOTS

There exist many algorithms for computing the roots of a polynomial:

- Bairstow, Graeffe, Jenkins-Traub, Laguerre, Müller, Newton, . . .

These methods yield satisfactory results if:

- The polynomial is of moderate degree
- The roots are simple and well-separated
- A good starting point in the iterative scheme is used

This heuristic has exceptions:

\[ f(x) = \prod_{i=1}^{20} (x - i) = (x - 1)(x - 2) \cdots (x - 20) \]
Example 1.1  Consider the polynomial
\[ x^4 - 4x^3 + 6x^2 - 4x + 1 = (x - 1)^4 \]
whose root is \( x = 1 \) with multiplicity 4. MATLAB returns the roots
\[ 1.0002, \quad 1.0000 + 0.0002i, \quad 1.0000 - 0.0002i, \quad 0.9998 \]

Example 1.2  The roots of the polynomial \((x - 1)^{100}\) were computed by MATLAB.

Figure 1.1: The computed roots of \((x - 1)^{100}\).
Figure 1.2: The root distribution of four perturbed polynomials.
Example 1.3

\[
\frac{\text{componentwise noise amplitude}}{\text{componentwise signal amplitude}} = 10^{-8}
\]

<table>
<thead>
<tr>
<th>exact root</th>
<th>multiplicity</th>
<th>computed root</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-6.7547000082e-1</td>
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</tr>
</tbody>
</table>

- The root multiplicities were calculated correctly
Example 1.4

\[
\frac{\text{componentwise noise amplitude}}{\text{componentwise signal amplitude}} = 10^{-8}
\]

<table>
<thead>
<tr>
<th>exact root</th>
<th>multiplicity</th>
<th>computed root</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

- The root multiplicities were calculated correctly
2. THE GEOMETRY OF ILL-CONDITIONED POLYNOMIALS

- A root $x_0$ of multiplicity $r$ introduces $(r - 1)$ constraints on the coefficients.
- A monic polynomial of degree $m$ has $m$ degrees of freedom.
- The root $x_0$ lies on a manifold of dimension $(m - r + 1)$ in a space of dimension $m$.
- This manifold is called a pejorative manifold because polynomials near this manifold are ill-conditioned.
- A polynomial that lies on a pejorative manifold is well-conditioned with respect to (the structured) perturbations that keep it on the manifold, which corresponds to the situation in which the multiplicity of the roots is preserved.
- A polynomial is ill-conditioned with respect to perturbations that move it off the manifold, which corresponds to the situation in which a multiple root breaks up into a cluster of simple roots.
Example 2.1 Consider a cubic polynomial $f(x)$ with real roots $x_0, x_1$ and $x_2$

$$(x - x_0)(x - x_1)(x - x_2) = x^3 - (x_0 + x_1 + x_2)x^2 + (x_0x_1 + x_1x_2 + x_2x_0)x - x_0x_1x_2$$

- If $f(x)$ has one double root and one simple root, then $x_0 = x_1 \neq x_2$ and thus $f(x)$ can be written as

  $$x^3 - (2x_1 + x_2)x^2 + (x_1^2 + 2x_1x_2)x - x_1^2x_2$$

  The pejorative manifold of a cubic polynomial that has a double root is the surface defined by

  $$\begin{pmatrix} -2x_1 - x_2 & x_1^2 + 2x_1x_2 & -x_1^2x_2 \end{pmatrix} \quad x_1 \neq x_2$$
• If \( f(x) \) has a triple root, then \( x_0 = x_1 = x_2 \) and thus \( f(x) \) can be written as

\[
x^3 - 3x_0x^2 + 3x_0^2x - x_0^3
\]

The pejorative manifold of a cubic polynomial that has a triple root is the curve defined by

\[
\left( -3x_0 \quad 3x_0^2 \quad -x_0^3 \right)
\]
Theorem 2.1  The condition number of the real root $x_0$ of multiplicity $r$ of the polynomial $f(x) = (x - x_0)^r$, such that the perturbed polynomial also has a root of multiplicity $r$, is

$$\rho(x_0) := \frac{\Delta x_0}{\Delta f} = \frac{1}{r |x_0|} \frac{\| (x - x_0)^r \|}{\| (x - x_0)^{r-1} \|} = \frac{1}{r |x_0|} \left( \frac{\sum_{i=0}^{r} \binom{r}{i}^2 (x_0)^{2i}}{\sum_{i=0}^{r-1} \binom{r-1}{i}^2 (x_0)^{2i}} \right)^{\frac{1}{2}}$$

where $\| \cdot \| = \| \cdot \|_2$ and

$$\Delta f = \frac{\| \delta f \|}{\| f \|} \quad \text{and} \quad \Delta x_0 = \frac{|\delta x_0|}{|x_0|}$$
Example 2.2  The condition number $\rho(1)$ of the root $x_0 = 1$ of $(x - 1)^r$ is

$$\rho(1) = \frac{1}{r} \left( \frac{\sum_{i=0}^{r} \binom{r}{i}^2}{\sum_{i=0}^{r-1} \binom{r-1}{i}^2} \right)^{\frac{1}{2}}$$

This expression reduces to

$$\rho(1) = \frac{1}{r} \sqrt{\frac{(2r)}{r^2}} = \frac{1}{r} \sqrt{\frac{2(2r - 1)}{r}} \approx \frac{2}{r} \quad \text{for large } r$$

Compare with the componentwise and normwise condition numbers

$$\kappa_c(1) \approx \frac{|\delta x_0|}{\varepsilon_c} \quad \text{and} \quad \kappa_n(1) \approx \frac{|\delta x_0|}{\varepsilon_n}$$

- $\rho(1)$ is independent of the the noise level (assumed to be small)
- $\rho(1)$ decreases as the multiplicity $r$ of the root $x_0 = 1$ increases
3. A SIMPLE POLYNOMIAL ROOT FINDER

Let \( w_i(x) \) be the product of all factors of degree \( i \) of \( f(x) \)

\[
f(x) = w_1(x)w_2^2(x)w_3^3(x) \cdots w_{r_{\text{max}}}^{r_{\text{max}}}(x)
\]

Perform a sequence of greatest common divisor (GCD) computations

\[
q_1(x) = \text{GCD}(f(x), f^{(1)}(x)) = w_2(x)w_3^2(x)w_4^3(x) \cdots w_{r_{\text{max}}^{r_{\text{max}}}}^{r_{\text{max}}-1}(x)
\]

\[
q_2(x) = \text{GCD}(q_1(x), q_1^{(1)}(x)) = w_3(x)w_4^2(x)w_5^3(x) \cdots w_{r_{\text{max}}}^{r_{\text{max}}-2}(x)
\]

\[
q_3(x) = \text{GCD}(q_2(x), q_2^{(1)}(x)) = w_4(x)w_5^2(x)w_6^3(x) \cdots w_{r_{\text{max}}}^{r_{\text{max}}-3}(x)
\]

\[
q_4(x) = \text{GCD}(q_3(x), q_3^{(1)}(x)) = w_5(x)w_6^2(x)w_7^3(x) \cdots w_{r_{\text{max}}}^{r_{\text{max}}-4}(x)
\]

\[
\vdots
\]

The sequence terminates at \( q_{r_{\text{max}}}(x) \), which is a constant.
A set of polynomials \( h_i(x), i = 1, \ldots, r_{\text{max}}, \) is defined such that

\[
\begin{align*}
h_1(x) &= \frac{f(x)}{q_1(x)} = w_1(x)w_2(x)w_3(x) \cdots \\
h_2(x) &= \frac{q_1(x)}{q_2(x)} = w_2(x)w_3(x) \cdots \\
h_3(x) &= \frac{q_2(x)}{q_3(x)} = w_3(x) \cdots \\
&\vdots \\
h_{r_{\text{max}}}(x) &= \frac{q_{r_{\text{max}}-2}}{q_{r_{\text{max}}-1}} = w_{r_{\text{max}}}(x)
\end{align*}
\]

The functions, \( w_1(x), w_2(x), \cdots, w_{r_{\text{max}}}(x), \) are determined from

\[
\begin{align*}
w_1(x) &= \frac{h_1(x)}{h_2(x)}, \quad w_2(x) = \frac{h_2(x)}{h_3(x)}, \quad \cdots, \quad w_{r_{\text{max}}-1}(x) = \frac{h_{r_{\text{max}}-1}(x)}{h_{r_{\text{max}}}(x)}
\end{align*}
\]

until

\[
w_{r_{\text{max}}}(x) = h_{r_{\text{max}}}(x)
\]
The equations

\[ w_1(x) = 0, \quad w_2(x) = 0, \quad \ldots, \quad w_{r_{\text{max}}}(x) = 0 \]

contain only simple roots, and they yield the simple, double, triple, etc., roots of \( f(x) \).

- If \( x_0 \) is a root of \( w_i(x) \), then it is a root of multiplicity \( i \) of \( f(x) \).

Mathematical operations performed in this root finder:

- GCD computations
- Polynomial divisions
- Solution of simple polynomial equations
Example 3.1 Calculate the roots of the polynomial

\[ f(x) = x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 \]

whose derivative is

\[ f^{(1)}(x) = 6x^5 - 15x^4 + 18x^2 - 6x - 3 \]

Perform a sequence of GCD computations

\[ q_1(x) = \text{GCD} \left( f(x), f^{(1)}(x) \right) = x^3 - x^2 - x + 1 \]
\[ q_2(x) = \text{GCD} \left( q_1(x), q_1^{(1)}(x) \right) = x - 1 \]
\[ q_3(x) = \text{GCD} \left( q_2(x), q_2^{(1)}(x) \right) = 1 \]

The maximum degree of a divisor of \( f(x) \) is 3 because the sequence terminates at \( q_3(x) \).
The polynomials $h_i(x)$ are:

\[
\begin{align*}
    h_1(x) &= \frac{f(x)}{q_1(x)} = x^3 - 2x^2 - x + 2 \\
    h_2(x) &= \frac{q_1(x)}{q_2(x)} = x^2 - 1 \\
    h_3(x) &= \frac{q_2(x)}{q_3(x)} = x - 1
\end{align*}
\]

The polynomials $w_i(x)$ are

\[
\begin{align*}
    w_1(x) &= \frac{h_1(x)}{h_2(x)} = x - 2 \\
    w_2(x) &= \frac{h_2(x)}{h_3(x)} = x + 1 \\
    w_3(x) &= h_3(x) = x - 1
\end{align*}
\]

and thus the factors of $f(x)$ are

\[f(x) = (x - 2)(x + 1)^2(x - 1)^3\]
3.1 Discussion of method

- The computation of the GCD of two polynomials is an ill-posed problem because it is not a continuous function of their coefficients:
  - The polynomials \( f(x) \) and \( g(x) \) may have a non-constant GCD, but the perturbed polynomials \( f(x) + \delta f(x) \) and \( g(x) + \delta g(x) \) may be coprime.

- The determination of the degree of the GCD of two polynomials reduces to the determination of the rank of a resultant matrix, but the rank of a matrix is not defined in a floating point environment.

- Polynomial division is an ill-posed problem:
  
  Even if \( \frac{f(x)}{g(x)} \) is a polynomial,
  
  \[ \frac{f(x) + \delta f(x)}{g(x) + \delta g(x)} \]
  is a rational function for arbitrary \( \delta f(x) \) and \( \delta g(x) \).
4. APPROXIMATE GREATEST COMMON DIVISORS

If \( f(x) \) is exact and all computations are performed in a symbolic environment, the GCD of \( f(x) \) and its derivative \( f^{(1)}(x) \) can be computed by the Sylvester resultant matrix \( S(f, f^{(1)}) \).

The polynomial \( f(x) \) is rarely known exactly, and so the given data is

\[
\tilde{f}(x) = f(x) + \delta f(x)
\]

and \( \tilde{f}(x) \) and \( \tilde{f}^{(1)}(x) \) are (with probability almost 1) coprime.

- The polynomials \( \tilde{f}(x) \) and \( \tilde{f}^{(1)}(x) \) have an approximate greatest common divisor (AGCD).

- Use the method of structured total least norm applied to \( S(\tilde{f}, \tilde{f}^{(1)}) \) to compute the smallest perturbation of \( S(\tilde{f}, \tilde{f}^{(1)}) \) such that its perturbed form is singular, which implies that the perturbed form \( \tilde{f}(x) \) of \( f(x) \) has a multiple root.
4.1 The Sylvester resultant matrix

The product of two polynomials is equal to the convolution of their coefficients:

\[
\begin{bmatrix}
  r_0 \\
  r_1 \\
  \vdots \\
  r_{m+n-1} \\
  r_{m+n}
\end{bmatrix}
= \begin{bmatrix}
  p_0 \\
  p_1 & p_0 \\
  \vdots & \ddots & \ddots \\
  p_m & \ddots & \ddots & p_0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  p_m & \ddots & \ddots & \ddots & \ddots & p_1 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & p_m
\end{bmatrix}
\begin{bmatrix}
  q_0 \\
  q_1 \\
  \vdots \\
  q_{n-1} \\
  q_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
  r_0 \\
  r_1 \\
  \vdots \\
  r_{m+n-1} \\
  r_{m+n}
\end{bmatrix}
= C_{n+1}(p) q = p \otimes q
\]

\[r \in \mathbb{R}^{m+n+1}, \ p \in \mathbb{R}^{m+1}, \ q \in \mathbb{R}^{n+1} \text{ and } C_{n+1}(p) \in \mathbb{R}^{(m+n+1) \times (n+1)}\]
Let:

- \( d_k(y) \) be a common divisor of degree \( k \) of the exact polynomials \( f(y) \) and \( f^{(1)}(y) \)
- The degree of the GCD of \( f(y) \) and \( f^{(1)}(y) \) be \( \hat{d} \)
- \( u_k(y) \) and \( v_k(y) \) be the quotient polynomials

\[
\begin{align*}
f(y) &= u_k(y)d_k(y) \\
f^{(1)}(y) &= v_k(y)d_k(y)
\end{align*}
\]

Thus

\[
f(y)v_k(y) - f^{(1)}(y)u_k(y) = 0 \iff C_{m-k}(f)v_k - C_{m-k+1}(f^{(1)})u_k = 0
\]

where

\[
\begin{align*}
C_{m-k}(f) &\in \mathbb{R}^{(2m-k)\times(m-k)} \\
C_{m-k+1}(f^{(1)}) &\in \mathbb{R}^{(2m-k)\times(m-k+1)} \\
v_k &\in \mathbb{R}^{m-k} \\
u_k &\in \mathbb{R}^{m-k+1}
\end{align*}
\]
\[
\begin{bmatrix}
C_{m-k}(f) & C_{m-k+1}(f^{(1)})
\end{bmatrix}
\begin{bmatrix}
v_k \\
-u_k
\end{bmatrix}
= S_k(f, f^{(1)})
\begin{bmatrix}
v_k \\
-u_k
\end{bmatrix}
= 0
\]

- \(S_k(f, f^{(1)}) \in \mathbb{R}^{(2m-k) \times (2m-2k+1)}\) and it is rank deficient
- The nullspace vectors yield the coefficients of the quotient polynomials
- Since the degree of the GCD of \(f(y)\) and \(f^{(1)}(y)\) is \(\hat{d}\), these polynomials possess common divisors of degrees \(1, 2, \ldots, \hat{d}\), but not a divisor of degree \(\hat{d} + 1\):

\[
\begin{align*}
\text{rank } S_k(f, f^{(1)}) &< 2m - 2k + 1, \quad k = 1, \ldots, \hat{d} \\
\text{rank } S_k(f, f^{(1)}) &= 2m - 2k + 1, \quad k = \hat{d} + 1, \ldots, m - 1
\end{align*}
\]

Calculating the degree of the GCD reduces to estimating the rank of a matrix.
Example 4.1 Consider $S_k(f, f^{(1)})$, for $k = 1, 2, 3$, for

$$f(x) = (x - 1)^2(x - 2)(x - 3) = x^4 - 7x^3 + 17x^2 - 17x + 6$$

$$f^{(1)}(x) = 4x^3 - 21x^2 + 34x - 17$$

Hence $S_1(f, f^{(1)}) = S(f, f^{(1)})$ is equal to

$$
\begin{bmatrix}
1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & -21 & 4 & 0 & 0 & 0 \\
17 & -7 & 1 & 34 & -21 & 4 & 0 & 0 \\
-17 & 17 & -7 & -17 & 34 & -21 & 4 & 0 \\
6 & -17 & 17 & 0 & -17 & 34 & -21 & 0 \\
0 & 6 & -17 & 0 & 0 & -17 & 34 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & -17 & 0 \\
\end{bmatrix}
$$

and this matrix has a unit loss of rank.
The subresultant matrix $S_2(f, f^{(1)})$ is

$$S_2(f, f^{(1)}) = \begin{bmatrix} 1 & 0 & 4 & 0 & 0 \\ -7 & 1 & -21 & 4 & 0 \\ 17 & -7 & 34 & -21 & 4 \\ -17 & 17 & -17 & 34 & -21 \\ 6 & -17 & 0 & -17 & 34 \\ 0 & 6 & 0 & 0 & -17 \end{bmatrix}$$

and this matrix has full column rank.
The subresultant matrix $S_3(f, f^{(1)})$ is

$$S_3(f, f^{(1)}) = \begin{bmatrix} 4 & 0 & 1 \\ -21 & 4 & -7 \\ 34 & -21 & 17 \\ -17 & 34 & -17 \\ 0 & -17 & 6 \end{bmatrix}$$

and this matrix has full column rank.

It follows that the first rank deficient matrix in the sequence

$$S_3(f, f^{(1)}), S_2(f, f^{(1)}), S_1(f, f^{(1)})$$

is $S_1(f, f^{(1)})$, and thus the degree of the GCD of $f(x)$ and $f^{(1)}(x)$ is one. □
4.2 Pre-processing operations for the computation of an AGCD

The computation of an AGCD of \( f(x) \) and \( f^{(1)}(x) \) requires that two pre-processing operations be performed:

- \( f(x) \) and \( f^{(1)}(x) \) must be normalised to balance the Sylvester matrix

- An AGCD of \( f(x) \) and \( f^{(1)}(x) \) is equal to, up to a scalar multiplier, an AGCD of \( f(x) \) and \( \alpha f^{(1)}(x) \), where \( \alpha \) is an arbitrary non-zero constant.

\[
\text{GCD} \left( f, f^{(1)} \right) \sim \text{GCD} \left( f, \alpha f^{(1)} \right), \quad \alpha \neq 0
\]

- The resultant matrix \( S(f, \alpha f^{(1)}) \) should be used when it is desired to compute an AGCD of \( f(x) \) and \( f^{(1)}(x) \)

- How is the optimal value of \( \alpha \) computed?
1. Normalisation: Define \( f(x) \) and \( g(x) \) as

\[
f(x) = \sum_{i=0}^{m} \bar{a}_i x^{m-i}, \quad \bar{a}_i = \frac{a_i}{\left(\prod_{j=0}^{m} |a_j|\right)^{\frac{1}{m+1}}}
\]

\[
g(x) = \sum_{i=0}^{m-1} b_i x^{m-1-i}, \quad \bar{b}_i = \frac{(m-i)\bar{a}_i}{\left(\prod_{j=0}^{m-1} |(m-j)\bar{a}_j|\right)^{\frac{1}{m}}}
\]

Note: \( g(x) \) is proportional to \( f^{(1)}(x) \)

2. The optimal value of \( \alpha \): Use linear programming to calculate \( \alpha_0 \), the optimal value of \( \alpha \)
4.3 The coprime polynomials and the degree of an AGCD

Recall that

\[
S_k(f, g) \begin{bmatrix} v_k \\ -u_k \end{bmatrix} = 0
\]

- \( S_k(f, g) \in \mathbb{R}^{(2m-k) \times (2m-2k+1)} \) and it is rank deficient
- The nullspace vectors yield the coefficients of the quotient polynomials
- If the degree of an GCD of \( f(x) \) and \( g(x) \) is \( d \), then

\[
\begin{align*}
\text{rank } S_k(f, g) &< 2m - 2k + 1, \quad k = 1, \ldots, d \\
\text{rank } S_k(f, g) &= 2m - 2k + 1, \quad k = d + 1, \ldots, m - 1
\end{align*}
\]

Use the same criterion for the calculation of the degree of an AGCD
• The degree $d$ of an AGCD of $f(x)$ and $g(x)$ is equal to the largest value of $k$, $k = 1, \ldots, m - 1$, such that $S_k(f, g)$ is numerically singular:

  - The SVD of $S_k(f, g)$ cannot be used because $f(x)$ and $g(x)$ are inexact and therefore, with high probability, coprime

  - The property

    \[
    \text{numerical rank } S_k(f, \alpha_0g) = \text{numerical rank } S_k \left( g, \frac{f}{\alpha_0} \right)
    \]

    enables a criterion for the calculation of $d$ to be developed

• $S_d(f, \alpha_0g)$ is numerically rank deficient by one and estimates of the coprime polynomials can be calculated from its nullspace

\[
S_d(f, \alpha_0g) \begin{bmatrix} v_d \\ -u_d \end{bmatrix} \approx 0
\]
5. STRUCTURED TOTAL LEAST NORM

Recall that \( S_d(f, \alpha_0 g) \) is numerically rank deficient by one and

\[
S_d(f, \alpha_0 g) \begin{bmatrix} v_d \\ -u_d \end{bmatrix} \approx 0
\]

If this equation is satisfied exactly, then

- The coprime polynomials are defined by the null space of \( S_d(f, \alpha_0 g) \)
- \( f(y) \) and \( g(y) \) have a non-constant common divisor:
  - \( f(y) \) has a multiple root
  - \( g(y) \) has been moved to a pejorative manifold
The calculation of \( d \) determines the column \( q \) of \( S_d(f, \alpha_0 g) \) such that if

\[
S_d(f, \alpha_0 g) = \begin{bmatrix} c_1 & c_1 & \cdots & c_{q-1} & c_q & c_{q+1} & \cdots & c_{2m-2k+1} \end{bmatrix}
\]

then the approximate homogeneous equation

\[
S_d(f, \alpha_0 g) \begin{bmatrix} \mathbf{v}_d \\ -\mathbf{u}_d \end{bmatrix} \approx 0
\]

can be transformed into an approximate linear algebraic equation

\[
\begin{bmatrix} c_1 & c_1 & \cdots & c_{q-1} & c_{q+1} & \cdots & c_{2m-2d+1} \end{bmatrix} \mathbf{x} \approx c_q
\]
or

\[ A_dx \approx c_q, \quad A_d \in \mathbb{R}^{(2m-d) \times (2m-2d)}, \quad c_q \in \mathbb{R}^{2m-d} \]

\[
x = \begin{bmatrix}
x_1 \\
\vdots \\
x_{q-1} \\
x_q - 1 \\
x_{q+1} \\
\vdots \\
x_{2m-2d}
\end{bmatrix} \in \mathbb{R}^{2m-2d}, \quad x_q = -1,
\]

\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_{q-1} \\
-1 \\
x_{q+1} \\
\vdots \\
x_{2m-2d}
\end{bmatrix} = \begin{bmatrix}
v_d \\
-u_d
\end{bmatrix}
\]
Recall:

- The degree of an AGCD of the given inexact polynomials $f(y)$ and $g(y)$ is $d$.
- The matrix $A_d$ and vector $c_q$ are functions of the coefficients of $f(y)$ and $g(y)$.

$$A_dx \approx c_q$$

is an approximate equation because its arguments are the coefficients of inexact polynomials.

Use structured total least norm to solve this approximate equation.
Given the inexact polynomials $f(y)$ and $g(y)$, which are assumed to be coprime, calculate the smallest perturbations that must be added to their coefficients such that the perturbed forms of $f(y)$ and $g(y)$ have a non-constant GCD.

**Aim:**

Compute the Sylvester matrix $S(\delta f, \alpha_0 \delta g)$, such that

$$\|\delta f\|^2 + \|\delta g\|^2$$

is minimised, where

$$S_d(f + \delta f, \alpha_0 (g + \delta g)) = S_d(f, \alpha_0 g) + S_d(\delta f, \alpha_0 \delta g)$$

is rank deficient.
The approximate under-determined equation

\[ A_d x \approx c_q \]

is corrected by considering the equation

\[ (A_d(\alpha_0) + E_d(\alpha_0, z)) x = c_q(\alpha_0) + h_q(\alpha_0, z) \]

which is non-linear in \( x \) and \( z \).

- \( A_d \) and \( E_d \) have the same structure, and \( c_d \) and \( h_d \) have the same structure:
  - Use the method of structured total least norm

- The initial vector of perturbations is \( z = 0 \)

- Solve this non-linear under-determined equation subject to the constraint that \( \|z\|^2 \) is minimised

- This leads to a least squares equality problem
The Sylvester matrix $S_d(f, \alpha_0 g)$ is

$$
\begin{bmatrix}
\bar{a} & \alpha_0 \bar{b} \\
\bar{a}_1 & \alpha_0 \bar{b}_1 & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\bar{a}_{m-1} & \bar{a}_1 & \alpha_0 \bar{b}_{n-1} & \ddots & \alpha_0 \bar{b}_1 \\
\bar{a}_m & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\bar{a}_{m-1} & \bar{a}_1 & \alpha_0 \bar{b}_{n-1} & \ddots & \ddots \\
\bar{a}_m & \ddots & \ddots & \ddots & \alpha_0 \bar{b}_n
\end{bmatrix}
$$
If the perturbations of the coefficients of $f(y)$ and $\alpha_0 g(y)$ are

$$z_i, \ i = 0, \ldots, m \quad \text{and} \quad \alpha_0 z_{m+1+i}, \ i = 0, \ldots, n$$

respectively, then $E_d(\alpha_0, z)$ is equal to

$$
\begin{bmatrix}
  z_0 & \alpha_0 z_{m+1} \\
  \vdots & \alpha_0 z_{m+2} & \ddots \\
  \vdots & \vdots & \ddots & \alpha_0 z_{m+1} \\
  z_{m-1} & \vdots & \alpha_0 z_{m+n} & \alpha_0 z_{m+2} \\
  z_m & \vdots & \alpha_0 z_{m+n+1} & \vdots \\
  \vdots & \vdots & \vdots & \alpha_0 z_{m+n} \\
  z_m & \alpha_0 z_{m+n+1}
\end{bmatrix}
$$
• Perform these AGCD computations repeatedly in order to determine the multiplicities of the roots
  – These calculations correspond to the identification of the pejorative manifold on which the theoretically exact polynomial lies

The other stages in the algorithm:

• Use the method of least squares to perform the polynomial division

• Recall it is necessary to solve the polynomial equations

\[ w_1(x) = 0, \quad w_2(x) = 0, \quad \cdots, \quad w_{r_{\text{max}}}(x) = 0 \]
  – Calculate the roots of the polynomial by solving these equations
6. EXAMPLES

Example 6.1

\[
\frac{\text{componentwise noise amplitude}}{\text{componentwise signal amplitude}} = 10^{-6}
\]

<table>
<thead>
<tr>
<th>exact root</th>
<th>multiplicity</th>
<th>computed root</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.10311000000e+0</td>
<td>2</td>
<td>8.1031311463e+0</td>
<td>3.8437554630e-6</td>
</tr>
<tr>
<td>3.50780000000e+0</td>
<td>8</td>
<td>3.5077983383e+0</td>
<td>4.7372726251e-7</td>
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<td>-6.30600000000e-1</td>
<td>8</td>
<td>-6.3060013449e-1</td>
<td>2.1327857935e-7</td>
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<tr>
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<td>9</td>
<td>-5.8210973315e+0</td>
<td>4.5841110328e-7</td>
</tr>
</tbody>
</table>

- The root multiplicities were calculated correctly
Example 6.2

\[
\frac{\text{componentwise noise amplitude}}{\text{componentwise signal amplitude}} = 10^{-8}
\]

<table>
<thead>
<tr>
<th>exact root</th>
<th>multiplicity</th>
<th>computed root</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>-7.3131318042e+0</td>
<td>9.3250329595e-6</td>
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<tr>
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<td>9.0182738917e+0</td>
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<td>6.6374090279e+0</td>
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<tr>
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<td>4</td>
<td>-1.9984000974e+0</td>
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<td>-8.7907151653e+0</td>
<td>1.7251509523e-6</td>
</tr>
</tbody>
</table>

- The root multiplicities were calculated correctly