Filters connecting Quadratic Systems

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Outline

- From strict equivalence to filters
- How to obtain filters?
- A parametrization of coprime filters
- Filters with nonsingular leading coefficient
- Stable filters

Classical Modal Analysis of Quadratic Systems

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$L(\lambda) = M\lambda^2 + D\lambda + K, \qquad M, D, K \in \mathbb{F}^{n\times n}$$

Decoupling by strict equivalence: $PL(\lambda) = L_D(\lambda)Q$ where

$$L_D(\lambda) = \mathrm{Diag}(p_1(\lambda), \ldots, p_n(\lambda)), \quad p_i(\lambda) = \lambda^2 + d_i\lambda + k_i \;\; (\det M \neq 0)$$

Condition (n. & s.) for decoupling: $DM^{-1}K = KM^{-1}D$ for

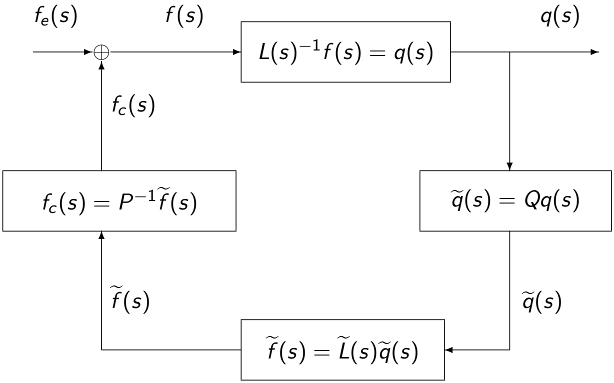
- Symmetric systems with $M > 0$ Caughey and O'Kelly [2], Bellman [1]
- Symmetric systems, $\lambda M + K$ semisimple and its real eigenvalues of definite type Lancaster and Z. [4]
- General systems and $\lambda M + K$ with distinct eigenvalues Ma and Caughey[5], Lancaster and Z.[4]

Few systems can be diagonalised by strict equivalence

Modal Control

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = f(t) \xrightarrow{\text{Laplace}} \overbrace{(Ms^2 + Ds + K)}^{L(s)} q(s) = f(s)$$

$$PL(s) = \widetilde{L}(s)Q$$



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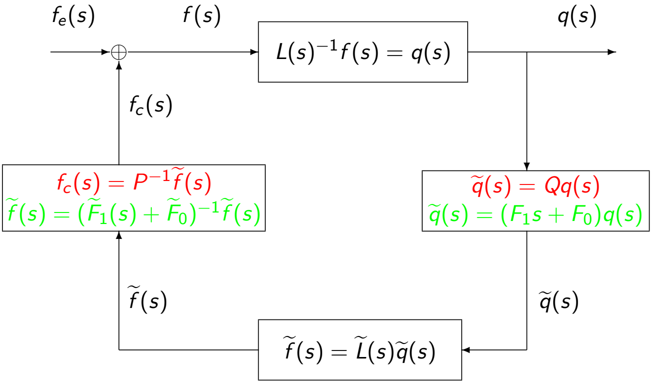

$$\widetilde{F}(s)L(s) = \widetilde{L}(s)F(s) \qquad F(s) \text{ and } \widetilde{F}(s) = \text{Filters}$$

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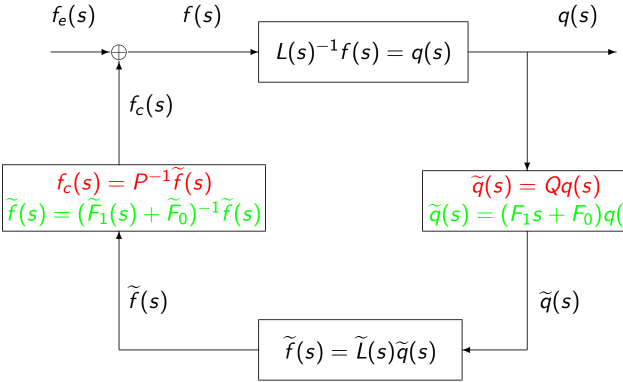

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$$f_c(s) = \widetilde{F}(s)^{-1}\widetilde{f}(s) \rightarrow \left(\widetilde{F}_1 \frac{d}{dt} + \widetilde{F}_0\right) f_c(t) = \widetilde{f}(t) \rightarrow \begin{cases} \det \widetilde{F}_1 \neq 0 \text{ and} \\ \widetilde{F}(s) \text{ stable} \end{cases}$$

Equivalence of Matrix Polynomials and Linearizations

$$P_1(\lambda) = U(\lambda)P_2(\lambda)V(\lambda),$$

$\det U(\lambda)$, $\det V(\lambda)$ constant $\neq 0$ polynomials.

Isospectral Systems= Equivalent Quadratic Matrix Polynomials

$A \in \mathbb{F}^{d\times d}$ **linearization** of $P(\lambda)$ if $\mathrm{Diag}(I, \lambda I_d - A)$ and $\mathrm{Diag}(I, P(\lambda))$ equivalent ($d = \deg(\det P(\lambda))$): $\lambda I_d - A$ and $P(\lambda)$ have the same elementary divisors or the same Jordan form.

Fundamental Theorem of Linear Algebra

If A_i linearization of $P_i(\lambda)$, $i = 1, 2$

$$P_1(\lambda) \overset{e}{\sim} P_2(s) \Leftrightarrow A_1 \overset{s}{\sim} A_2$$

Equivalence leads to filters

$$\widetilde{L}(\lambda)V(\lambda) = U(\lambda)L(\lambda) \tag{1}$$

$$U(\lambda) = \widetilde{L}(\lambda)\widetilde{Q}(\lambda) + \widetilde{F}(\lambda), \quad V(\lambda) = Q(\lambda)L(\lambda) + F(\lambda)$$

$\widetilde{L}(\lambda)^{-1}\widetilde{F}(\lambda)$ and $F(\lambda)L(\lambda)^{-1}$ strictly proper (or zero)

Substitute in (1):

$$\widetilde{L}(\lambda)(Q(\lambda)L(\lambda) + F(\lambda)) = (\widetilde{L}(\lambda)\widetilde{Q}(\lambda) + \widetilde{F}(\lambda))L(\lambda)$$
$$\Updownarrow$$
$$\widetilde{L}(\lambda)(Q(\lambda) - \widetilde{Q}(\lambda))L(\lambda) = \widetilde{F}(\lambda)L(\lambda) - \widetilde{L}(\lambda)F(\lambda)$$

Then

$$Q(\lambda) - \widetilde{Q}(\lambda) = \widetilde{L}(\lambda)^{-1}\widetilde{F}(\lambda) - F(\lambda)L(\lambda)^{-1} = 0$$

and

$$\widetilde{F}(\lambda)L(\lambda) = \widetilde{L}(\lambda)F(\lambda)$$

Using Rosenbrock's[2] polynomial methods

$L(\lambda)$, $F(\lambda)$ right coprime and $\widetilde{L}(\lambda)$, $\widetilde{F}(\lambda)$ left coprime

[2] State-Space and Multivariable Theory. Thomas Nelson and Sons, 1971

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Filters and Equivalence

Theorem

$L(\lambda) = M\lambda^2 + D\lambda + K$ y $\widetilde{L}(\lambda) = \widetilde{M}\lambda^2 + \widetilde{D}\lambda + \widetilde{K}$ are equivalent if and only if there are linear matrix polynomials $F(\lambda) = F_1\lambda + F_0$ and $\widetilde{F}(\lambda) = \widetilde{F}_1\lambda + \widetilde{F}_0$ such that

(i) *$L(\lambda)^{-1}F(\lambda)$ and $\widetilde{F}(\lambda)\widetilde{L}(\lambda)^{-1}$ are strictly proper rational function matrices*

(ii) *$L(\lambda)$, $F(\lambda)$ are right coprime, and $\widetilde{L}(\lambda)$, $\widetilde{F}(\lambda)$ are left coprime*[a]

(iii) $\widetilde{F}(\lambda)L(\lambda) = \widetilde{L}(\lambda)F(\lambda)$

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[a] P. A. Fuhrmann: On strict system equivalence and similarity. Int. J. Control (1977), 25 (1), 5-10.

- (iii) $\Rightarrow$ $F(\lambda)$ and $\widetilde{F}(\lambda)$ are filters from $L(\lambda)$ to $\widetilde{L}(\lambda)$.
- If $F(\lambda)$ and $\widetilde{F}(\lambda)$ satisfy (i)-(iii): Strictly proper coprime filters.
- If $\det M \neq 0$ and $\det \widetilde{M} \neq 0$, (i) always holds true.
- If $F(\lambda)$ and $\widetilde{F}(\lambda)$ are strictly proper coprime filters from $L(\lambda)$ to $\widetilde{L}(\lambda)$ then they are equivalent.

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Controllability, Observability and Minimal Realizations

$A \in \mathbb{F}^{n\times n}, B \in \mathbb{F}^{n\times m}, C \in \mathbb{F}^{p\times n}$

- (A, B) controllable $\mathrm{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$
- (C, A) observable $\mathrm{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

If $G(\lambda)$ is a matrix of strictly proper rational functions, then there always exist A, B and C such that $G(\lambda) = C(\lambda I_d - A)^{-1}B$.

- (A, B, C)= realization of $G(\lambda)$
- d= order of the realization

(A, B, C) minimal realization if d is as small as possible $\Leftrightarrow$ (A, B) controllable and (C, A) observable.

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Controllable and Observable Realizations of Matrix Polynomials

$L(\lambda)$ $n \times n$ matrix polynomial with $\deg(\det L(\lambda)) = d$

Definition

- (A, Y) controllable realization of $L(\lambda)$ if Y full rank $d \times n$ and
$$N(\lambda)L(\lambda)^{-1} = (\lambda I_d - A)^{-1} Y$$
- (X, A) observable realization of $L(\lambda)$ if X full rankf $n \times d$ and
$$L(\lambda)^{-1}N(\lambda) = X(\lambda I_d - A)^{-1}$$

There always exist controllable and observable realizations of $L(\lambda)$

If $L(\lambda)$ and $\widetilde{L}(\lambda)$ are isospectral there are X, A and Y such that (Y, A) is a controllable realization of $L(\lambda)$ and (X, A) is an observable realization of $\widetilde{L}(\lambda)$

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Obtaining coprime filters

$$N(\lambda)L(\lambda)^{-1} = (\lambda I_d - A)^{-1}Y \Leftrightarrow (\lambda I_d - A)N(\lambda) = YL(\lambda)$$
$$\Rightarrow N(\lambda) = N_1\lambda + N_0, \begin{cases} N_1 = YM \\ N_0 = YD + AYM \end{cases}$$

If $L(\lambda)$ and $\widetilde{L}(\lambda)$ isospectral, $N(\lambda)L(\lambda)^{-1} = (\lambda I - A)^{-1}Y$ and $\widetilde{L}(\lambda)^{-1}\widetilde{N}(\lambda) = X(\lambda I - A)^{-1}$ then

$$X(\lambda I - A)^{-1}Y = \underbrace{XN(\lambda)}_{F(\lambda)}L(\lambda)^{-1} = \widetilde{L}(\lambda)^{-1}\underbrace{\widetilde{N}(\lambda)Y}_{\widetilde{F}(\lambda)}$$

$$\widetilde{L}(\lambda)F(\lambda) = \widetilde{F}(\lambda)L(\lambda)$$

$F(\lambda)$ and $\widetilde{F}(\lambda)$ strictly proper coprime filters from $L(\lambda)$ to $\widetilde{L}(\lambda)$ and

$$F(\lambda) = XYM\lambda + (XYD + XAYM), \quad \widetilde{F}(\lambda) = \widetilde{M}XY\lambda + (\widetilde{D}XY + \widetilde{M}XAY)$$

And conversely

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Diagonalizable systems and Krylov subspaces

$L(\lambda) = M\lambda^2 + D\lambda + K$, $\widetilde{L}(\lambda) = \mathrm{Diag}(p_i(\lambda))$, $p_i(\lambda) = \lambda^2 + d_i\lambda + k_i$

Linearizations

$$C_L = \begin{bmatrix} 0 & -KM^{-1} \\ I_n & -DM^{-1} \end{bmatrix}; \quad A_L = \mathrm{Diag}(A_{Li}), A_{Li} = \begin{bmatrix} 0 & -k_i \\ 1 & -d_i \end{bmatrix}$$

$$L(\lambda) \text{ and } \widetilde{L}(\lambda) \text{ isospectral}$$
$$\Updownarrow$$
$$\exists T \in \mathrm{Gl}_{2n}(\mathbb{F}) \text{ s. t. } C_L T = T A_L$$
$$\Updownarrow$$
$$T = \begin{bmatrix} b_1 & C_L b_1 & b_2 & C_L b_2 & \cdots & b_n & C_L b_n \end{bmatrix}$$
$$\text{and } C_L^2 b_i + d_i C_L b_i + k_i b_i = p_i(C_L) b_i = 0$$
$$\Updownarrow$$
$$\mathbb{F}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

$V_i =< b_i, C_L b_i >$ a C_L-cyclic (Krylov) subspace of dimension 2 with minimal polynomial $p_i(\lambda)$.

Filters for diagonalizable systems I

$$X = \begin{bmatrix} 0 & M^{-1} \end{bmatrix}, Y_0 = \mathrm{Diag}(Y_1, \dots, Y_n), Y_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- (A_L, Y_0) controllable: $\mathrm{rank}\begin{bmatrix} Y_0 & A_L Y_0 & \cdots & A_L^{2n-1} Y_0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} y_1 & A_L y_1 & \cdots & y_n & A_L y_n \end{bmatrix} = \mathrm{rank}\, I_{2n} = 2n$
- (X, C_L) observable: $\mathrm{rank}\begin{bmatrix} X \\ XC_L \\ \vdots \\ XC_L^{2n-1} \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 0 & M^{-1} \\ M^{-1} & * \end{bmatrix} = 2n$

$$L(\lambda)X = \begin{bmatrix} 0 & \lambda^2 I_n + \lambda D M^{-1} + K M^{-1} \end{bmatrix} = \overbrace{\begin{bmatrix} I_n & \lambda I_n \end{bmatrix}}^{N(s)} \overbrace{\begin{bmatrix} \lambda I_n & K M^{-1} \\ -I_n & \lambda I_n + D M^{-1} \end{bmatrix}}^{(\lambda I_{2n} - C_L)}$$

$$Y_i p_i(\lambda) = \begin{bmatrix} \lambda^2 + d_i \lambda + k_i \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda & k_i \\ -1 & \lambda + d_i \end{bmatrix} \begin{bmatrix} \lambda + d_i \\ 1 \end{bmatrix} = (\lambda I_2 - A_{Li}) \widetilde{N}_i(\lambda)$$

If $\widetilde{N}(\lambda) = \mathrm{Diag}(\widetilde{N}_i(\lambda))$
$X(\lambda I_{2n} - C_L)^{-1} = L(\lambda)^{-1} N(\lambda)$ and $(\lambda I_{2n} - A_L)^{-1} Y_0 = \widetilde{N}(\lambda) \widetilde{L}(\lambda)^{-1}$

Filters for diagonalizable systems II

$A_L = T^{-1}C_L T$, $T = \begin{bmatrix} b_1 & C_L b_1 & \cdots & b_n & C_L b_n \end{bmatrix}$:

$$(\lambda I_{2n} - C_L)^{-1} T Y_0 = T\widetilde{N}(\lambda)\widetilde{L}(\lambda)^{-1}$$

and
$$X(\lambda I_{2n} - C_L)^{-1} T Y_0 = \begin{cases} (XT\widetilde{N}(\lambda))\widetilde{L}(\lambda)^{-1} = \widetilde{F}(\lambda)\widetilde{L}(\lambda)^{-1} \\ L(\lambda)^{-1}(N(\lambda)TY_0) = L(\lambda)^{-1}F(\lambda) \end{cases}$$

(X, C_L)=observable realization of $L(\lambda)$, (C_L, TY_0)=controllable realization of $\widetilde{L}(\lambda)$

$$L(\lambda)\widetilde{F}(\lambda) = F(\lambda)\widetilde{L}(\lambda)$$

$TY_0 = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$. If $b_i = \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix}$:

For $i = 1, 2, \ldots, n$, $L(\lambda)(v_{i1}\lambda + v_{i2}) = (\lambda^2 + d_i\lambda + k_i)(b_{i2}\lambda + b_{i1})$ where $v_{i1} = M^{-1}b_{i2}$ and $v_{i2} = M^{-1}((d_i I_n - DM^{-1})b_{i2} + b_{i1})$.

Analogy

If (λ_0, v) is an eigenpair of $\lambda A + B$

$$(\lambda_0 A + B)v = 0 \Leftrightarrow (\lambda A + B)v = (\lambda - \lambda_0)(Av)$$

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$$(\lambda_0 A + B)v = 0 \Leftrightarrow (\lambda A + B)v = (\lambda - \lambda_0)(Av)$$

Example

$$L(\lambda)=\lambda^2\begin{bmatrix}3&1\\1&2\end{bmatrix}+\lambda\begin{bmatrix}3&6\\9&3\end{bmatrix}+\begin{bmatrix}6&2\\2&4\end{bmatrix},\widetilde{L}(\lambda)=\begin{bmatrix}\lambda^2-1&0\\0&\lambda^2-4\end{bmatrix}$$

$$\Lambda(C_L)=\{1,2,-2,-1\}.\ \text{Eigenvectors: }\begin{bmatrix}2\\-2\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}2\\2\\1\\1\end{bmatrix}$$

$$p_1(\lambda)=(\lambda+1)(\lambda-1)\to b_1=\begin{bmatrix}2\\-2\\-1\\1\end{bmatrix}+\begin{bmatrix}2\\2\\1\\1\end{bmatrix}=\begin{bmatrix}4\\0\\0\\2\end{bmatrix}$$

$$p_2(\lambda)=(\lambda+2)(\lambda-2)\to b_2=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\0\\0\\2\end{bmatrix}$$

$$b_{12}\lambda+b_{11}=\begin{bmatrix}0\\2\end{bmatrix}\lambda+\begin{bmatrix}4\\0\end{bmatrix},\quad b_{22}\lambda+b_{21}=\begin{bmatrix}0\\2\end{bmatrix}\lambda+\begin{bmatrix}2\\0\end{bmatrix}$$

$$F(\lambda)=\begin{bmatrix}4&2\\2\lambda&2\lambda\end{bmatrix},\widetilde{F}(\lambda)=\begin{bmatrix}-2/5\lambda-4/5&-2/5\lambda-8/5\\6/5\lambda+2/5&6/5\lambda+4/5\end{bmatrix}$$

$$L(\lambda)\widetilde{F}(\lambda)=F(\lambda)\widetilde{L}\lambda)$$

Stable filters with Regular leading coefficient

For any two isospectral systems:

Are there always STABLE filters with REGULAR leading coefficient?

$F(\lambda) = F_1\lambda + F_0$ STRONGLY REGULAR if $\det F_1 \neq 0$
$F(\lambda)$ stable if its eigenvalues are placed in a prescribed region of the complex plane.

How to obtain all filters?

Parametrizing filters

$$\mathcal{T} = \{(X, A, Y) : (X, A) \text{ obs. real. } \widetilde{L}(\lambda), (A, Y) \text{ cont. real. } L(\lambda)\}$$

$$F(\lambda) = XYM\lambda + (XYD + XAYM), \quad \widetilde{F}(\lambda) = \widetilde{M}XY\lambda + (\widetilde{D}XY + \widetilde{M}XAY)$$

Similar triples $((X, A, Y) \to (XP, P^{-1}AP, P^{-1}Y))$ in $\mathcal{T}$ produce the same filters.

Conclusion: If $\mathcal{F}$ is the set of all strictly proper coprime filters from $L(\lambda)$ to $\widetilde{L}(\lambda)$, then the mapping

$$\varphi : \mathcal{T}/\,\mathrm{Gl} \longrightarrow \mathcal{F}$$

given by

$$\varphi([X, A, Y)]) = (XYM\lambda + (XYD + XAYM), \widetilde{M}XY\lambda + (\widetilde{D}XY + \widetilde{M}XAY))$$

is a bijection.

A better parametrization?

Given $(X_0, A_0, Y_0) \in \mathcal{T}$, let $Z(A_0) = \{H : HA_0 = A_0 H\}$ and

$$Z^*(A_0) = \{H \text{ invertible } : HA_0 = A_0 H\}$$

Theorem

$$\psi : Z^*(A_0) \longrightarrow \mathcal{F}$$

given by

$$\psi(H) = (F(\lambda), \widetilde{F}(\lambda))$$

$$F(\lambda) = X_0 H Y_0 M \lambda + (X_0 H Y_0 D + X_0 H A_0 Y_0 M),$$
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is a bijection.

Natural choices of A_0 for diagonalizable systems:

- complex or real Jordan form
- block-diagonal companion of the diagonal system

A better parametrization?

Given $(X_0, A_0, Y_0) \in \mathcal{T}$, let $Z(A_0) = \{H : HA_0 = A_0 H\}$ and

$$Z^*(A_0) = \{H \text{ invertible } : HA_0 = A_0 H\}$$

Theorem

$$\psi : Z^*(A_0) \longrightarrow \mathcal{F}$$

given by

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Existence of Strongly Regular Filters

$$F(\lambda) = XHYM\lambda + (XHYD + XHAYM), \widetilde{F}(\lambda) = \widetilde{M}XHY\lambda + (\widetilde{D}XHY + \widetilde{M}XHAY)$$

If $\det M = 0$ no strictly proper coprime strongly regular filters

If $\det M \neq 0$ "almost all" strictly proper coprime filters are strongly regular

"Almost all" = all but a set defined as the solution of a finite number of polynomial equations.

Set of strictly proper coprime filters with regular leading coefficient can be identified with

$$\mathcal{C} = \{H \in Z^*(A) : \det(XHY) \neq 0\}$$

(X, A) obs. real. of $\widetilde{L}(\lambda)$ and (A, Y) cont. real. of $L(\lambda)$.

If $\mathcal{C} \neq \emptyset$ then it is open and dense in $Z^*(A)$.

$\boxed{\mathcal{C} \neq \emptyset ?}$

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Strongly regular filters for diagonalizable systems I

$$L(\lambda) = \lambda^2 \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 & 6 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, \widetilde{L}(\lambda) = \begin{bmatrix} \lambda^2 - 1 & 0 \\ 0 & \lambda^2 - 4 \end{bmatrix}$$

$$C_R = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, A_R = \mathrm{Diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \right)$$

$$X = \mathrm{Diag}\left(\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right), Y_0 = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

(X, A_R)=observable realiz. of $\widetilde{L}(\lambda)$, (C_R, Y_0)= controllable realiz. of $L(\lambda)$

If $T^{-1}C_R T = A_R$ then (A_R, Y)= controllable realiz. of $L(\lambda)$, $Y = T^{-1}Y_0$

$$\widetilde{F}(\lambda) = \widetilde{M}XY\lambda + \widetilde{D}XY + \widetilde{M}XA_R Y = \begin{bmatrix} 3/5 & -1/5 \\ -3/5 & 1/5 \end{bmatrix} \lambda + \begin{bmatrix} 1/5 & -3/5 \\ -2/5 & 6/5 \end{bmatrix}$$

$$F(\lambda) = XYM\lambda + XYD + XA_R YM = \begin{bmatrix} 8/5 & 1/5 \\ -8/5 & -1/5 \end{bmatrix} \lambda + \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

Strongly Regular filters for diagonalizable systems II

$$H \in Z(A_R) \Leftrightarrow H = \mathrm{Diag}(H_1, H_2), H_1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, H_2 = \begin{bmatrix} c & d \\ 4d & c \end{bmatrix},$$

$$XHY = \begin{bmatrix} a & b & & \\ & & c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} a & \\ & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \end{bmatrix} + \begin{bmatrix} b & \\ & d \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix}$$

$$\begin{aligned} \widetilde{F}(\lambda) &= \widetilde{M}XHY\lambda + \widetilde{D}XHY + \widetilde{M}HXA_RY \\ &= \begin{bmatrix} \frac{3a+b}{5} & -\frac{a+3b}{5} \\ -\frac{3c+2d}{5} & \frac{c+6d}{5} \end{bmatrix} \lambda + \begin{bmatrix} \frac{a+3b}{5} & -\frac{3a+b}{5} \\ -\frac{2c+12d}{5} & \frac{6c+4d}{5} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} F(\lambda) &= XHYM\lambda + XHYD + XHA_RYM \\ &= \begin{bmatrix} \frac{8a}{5} & \frac{a-5b}{5} \\ -\frac{8c}{5} & \frac{10d-c}{5} \end{bmatrix} \lambda + \begin{bmatrix} -\frac{16b}{5} & \frac{10a-2b}{5} \\ \frac{16d}{5} & \frac{2d-5c}{5} \end{bmatrix} \end{aligned}$$

$$\det F_1 = \frac{16ad - 8bc}{5}, \quad \det \widetilde{F}_1 = \frac{16ad - 8bc}{25}$$

$$\{H \in Z^*(A_R) : \det(XHY) \neq 0\} \equiv$$
$$\{(a,b,c,d) \in \mathbb{F}^4 : (a^2 - b^2) \neq 0, (c^2 - 4d^2) \neq 0, (2ad - bc) \neq 0\}$$

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$$\{(a, b, c, d) \in \mathbb{F}^4 : (a^2 - b^2) \neq 0, (c^2 - 4d^2) \neq 0, (2ad - bc) \neq 0\}$$

Stable filters for diagonalizable systems

Eigenvalue placement may depend on the field:

$$L(\lambda) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \lambda^2 + \begin{bmatrix} -9 & -3 \\ -3 & 4 \end{bmatrix} \lambda + \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, \widetilde{L}(\lambda) = \begin{bmatrix} \lambda^2 - 3\lambda + 2 & 0 \\ 0 & \lambda^2 + 3\lambda + 2 \end{bmatrix}$$

General form of filters:

$$F(\lambda) = \begin{bmatrix} b & b/3 \\ 0 & d \end{bmatrix} \lambda + \begin{bmatrix} a & a/3 \\ 0 & c \end{bmatrix}$$

$$\widetilde{F}(\lambda) = \begin{bmatrix} b/3 & 0 \\ -d/5 & 3d/5 \end{bmatrix} \lambda + \begin{bmatrix} a/3 & 0 \\ -c/5 & 3c/5 \end{bmatrix}$$

If $H \in Z(A_R)$ then $\det H = (a+b)(a+2b)(c-d)(c-2d)$

$$\begin{bmatrix} a/3 & 0 \\ -c/5 & 3c/5 \end{bmatrix} L(\lambda) = \widetilde{L}(\lambda) \begin{bmatrix} a & a/3 \\ 0 & c \end{bmatrix}$$

Stability as an inverse eigenvalue problem on pencils

Assume $L(\lambda)$ and $\widetilde{L}(\lambda)$ are isospectral systems, (X, A) an observable realization of $\widetilde{L}(\lambda)$ and (A, Y) controllable realization of $L(\lambda)$. Then there are strictly proper coprime filters $F(\lambda)$ and $\widetilde{F}(\lambda)$:

$$X(\lambda I - A)^{-1} Y = F(\lambda) L(\lambda)^{-1} = \widetilde{L}(\lambda)^{-1} \widetilde{F}(\lambda)$$

Irreducible matrix descriptions of the same rational matrix.

Rosenbrock's theory:

- $(\lambda I - A)$, $L(\lambda)$ and $\widetilde{L}(\lambda)$ have the same elementary divisors
- $F(\lambda)$, $\widetilde{F}(\lambda)$ and $\begin{bmatrix} \lambda I - A & Y \\ -X & 0 \end{bmatrix}$ have the same elementary divisors.

Problem

Given (X, A, Y) with (X, A) an observable realization of $\widetilde{L}(\lambda)$ and (A, Y) a controllable realization of $L(\lambda)$, characterize

$$\Lambda \left\{ \begin{bmatrix} \lambda I - A & Y \\ -XH & 0 \end{bmatrix} : H \in Z^*(A) \right\}$$

Conclusions

- Stable Filters with regular leading coefficients can be used for modal control of Quadratic Systems
- Equivalence of Quadratic Systems can be given through strictly proper coprime filters
- Basic control theory can be used to obtain strictly proper coprime filters
- The set of strictly proper coprime filters between two systems can be parametrized with the centralizer of any linearization as a parametrization space.
- If the mass matrix is singular there is no strictly proper coprime filters with regular leading coefficients
- If the mass matrix is regular almost all strictly proper coprime filters have regular leading coefficients
- The characterization of the Smith invariants of filters is a big usue that remains unsolved

References

R. Bellman.
Introduction to Matrix Analysis
SIAM, 1995

T. K. Caughey, M.E. J. O'Kelly.
Classical Normal Modes in Damped Linear Dynamic Systems
ASME JOURNAL OF APPLIED MECHANICS, Vol. 32, (1965) pp. 583–588.

S. Garvey, P. Lancaster, A. Popov, U. Prells, I. Zaballa
Filters connecting quadratic systems. Part 2
In preparation.

P. Lancaster, I. Zaballa
Diagonalizable quadratic eigenvalue problems
Mechanical Systems and Signal Processing Vol. 23 (2009) pp. 1134–1144.

F. Ma, T. K. Caughey.
Analysis of Linear Nonconservative Vibrations
ASME JOURNAL OF APPLIED MECHANICS, Vol. 22, (1995) pp. 685–691.