An inverse problem of Calderón type with partial data

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Graz University of Technology

jointly with Jonathan Rohleder

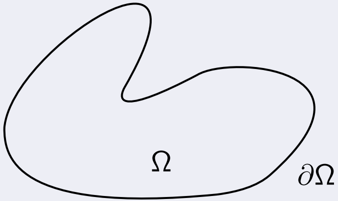
MOPNET 5 meeting, UCL

PART I

The Calderón problem (with partial data)

Calderón's problem

$\Omega \subseteq \mathbb{R}^n$ bdd., $\partial\Omega$ C^∞
$\gamma : \overline{\Omega} \to \mathbb{R}$, $\gamma \geq \epsilon > 0$



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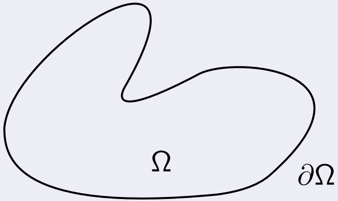

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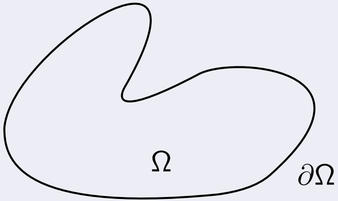

$\nabla \cdot \gamma \nabla u = 0, \; u|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega)$

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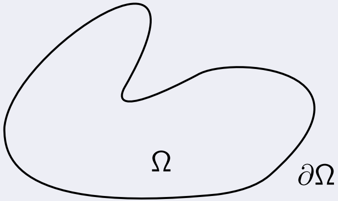

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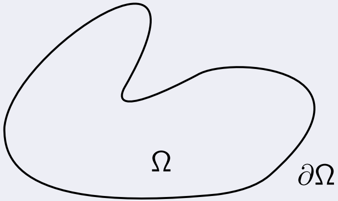

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Dirichlet-to-Neumann map

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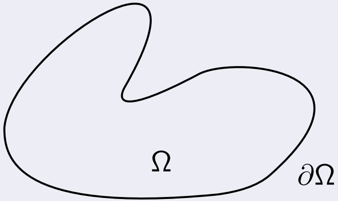

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- Uniqueness: $M_{\gamma_1} = M_{\gamma_2} \overset{?}{\Longrightarrow} \gamma_1 = \gamma_2$ in Ω

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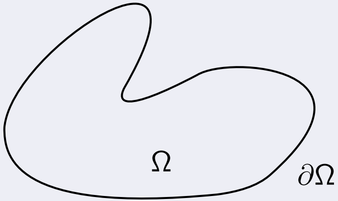

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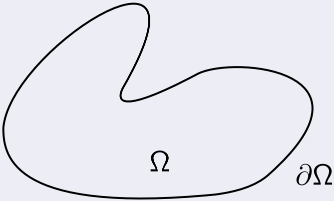

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Origins: EIT; γ isotropic conductivity of inhom. body Ω

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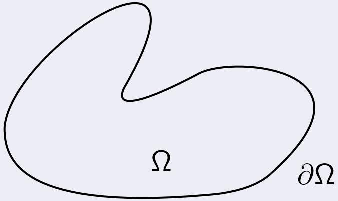

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Theorem [SylvesterUhlmann, 1987]

Let $n \geq 3$ and $\gamma_1, \gamma_2 \in C^\infty(\overline{\Omega})$. If

$$M_{\gamma_1} f = M_{\gamma_2} f, \quad f \in H^{1/2}(\partial\Omega) \text{ on } \partial\Omega$$

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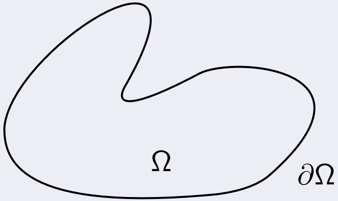

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Theorem [NachmanSylvesterUhlmann, 1988]

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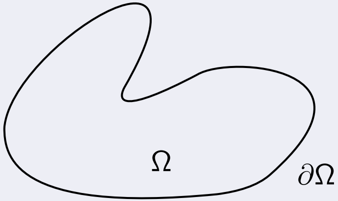

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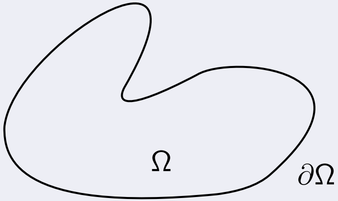

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Theorem [Nachman, 1996]

Let $n = 2$ and $\gamma_1, \gamma_2 \in W^{2,p}(\Omega)$ for some $p > 1$. If

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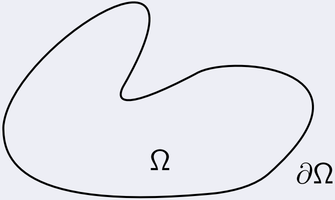

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Theorem [AstalaPäivärinta, 2006]

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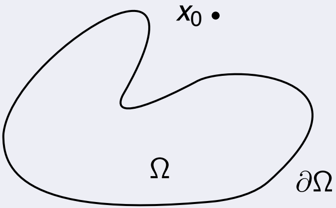

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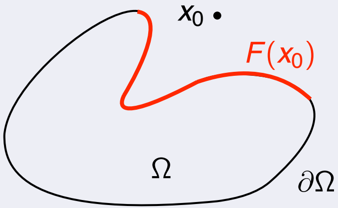

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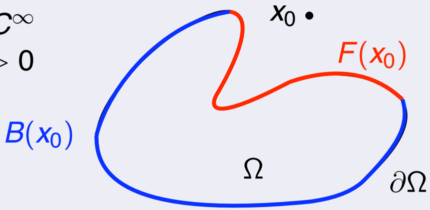

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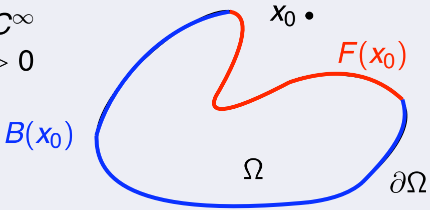

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Theorem [KenigSjöstrandUhlmann, 2007]

Let $n \geq 3$, $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$, $\widetilde{F} \supseteq F(x_0)$, $\widetilde{B} \supseteq B(x_0)$, $\widetilde{F}, \widetilde{B} \subset \partial\Omega$,

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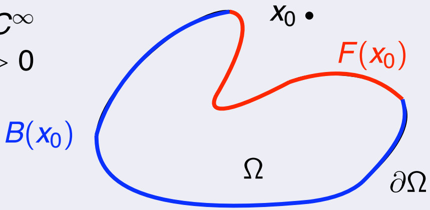

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$$\{\partial\mathcal{M}, \lambda_j, \partial_\nu \varphi_j|_{\partial\mathcal{M}}\}_{j=1}^{\infty} \Rightarrow \{\mathcal{A}, \mathcal{M}\} \text{ up to gauge equiv.}$$

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  - Some milestones [Belishev, 1987] [BelishevKurylev, 1992]

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- Inverse problem for the wave equation with elliptic data or boundary spectral data

PART II

An inverse problem of Calderón type

with partial data

Calderón type problem with partial data

- Instead $\nabla \cdot \gamma \nabla$ consider

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a$$

unif. elliptic on $\overline{\Omega}$, $a_{jk}, a_j \in C^{0,1}(\overline{\Omega})$, $a \in L^\infty(\Omega, \mathbb{R})$, $a_{jk} = \overline{a_{kj}}$

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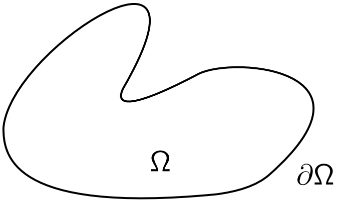

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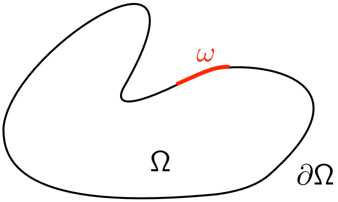

- ω (arbitrarily small) open subset of $\partial\Omega$

Selfadjoint Dirichlet operator in $L^2(\Omega)$

$$Au = \mathcal{L}u, \quad \operatorname{dom} A = \{u \in H_0^1(\Omega) : \mathcal{L}u \in L^2(\Omega)\}$$

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λ-dependent Dirichlet-to-Neumann map

For $\lambda \in \rho(A)$ define

$$M(\lambda) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad u_\lambda|_{\partial\Omega} \mapsto \frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega},$$

with $u_\lambda \in H^1(\Omega)$ unique sol. of $\mathcal{L}u = \lambda u$, $u_\lambda|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega)$

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ATTENTION !

a_{jk}, a_j, a are **NOT** uniquely determined by $M(0)$ or $M(\lambda)$

A mild uniqueness result

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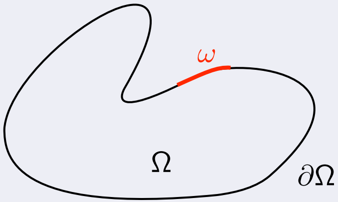
$n = 2, 3, \ldots$



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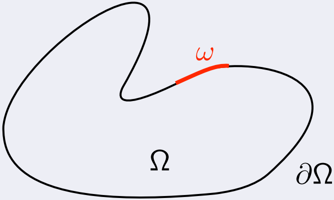

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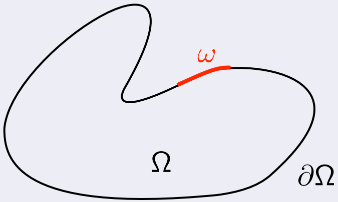

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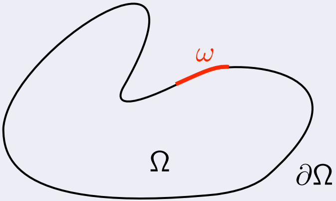

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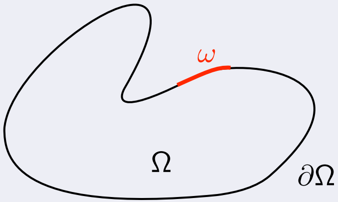

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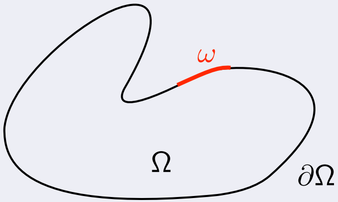

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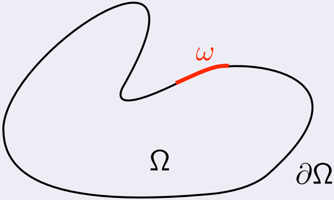

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$$Au = \sum_{k=1}^{\infty} \mu_k (u, \psi_k)\psi_k, \quad u \in \operatorname{dom} A.$$

PART III

What is behind ?

Ideas and methods

Extension theory of symmetric operators

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Unique continuation

[Aronszjan, 1957][Hörmander, 1983][Wolff, 1993][Tataru, 1995]

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- Weyl or M-function of T: DN-map $M_\omega(\cdot)$ on ω

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B. & Rohleder, *An inverse problem of Calderón type with partial data*, to appear in Comm. PDE

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