Electromagnetic fields of charged beams in gradually tapering waveguides

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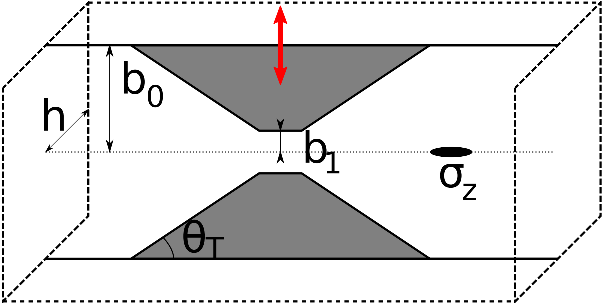
Motivation

Examples of non-uniform metallic structures in accelerators :

- collimators



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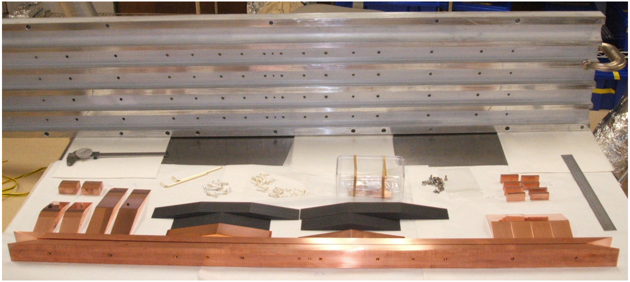
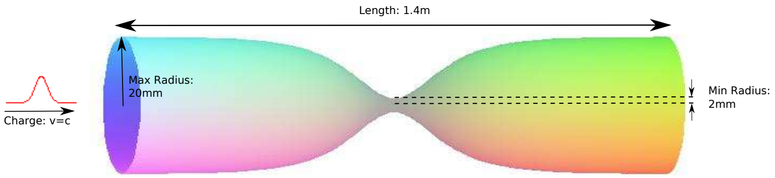


Motivation

Examples of non-uniform metallic structures in accelerators :

- transition between subsystems, e.g. beam pipe in a small gap undulator



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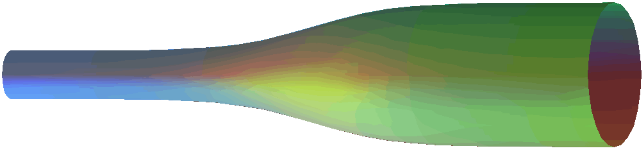

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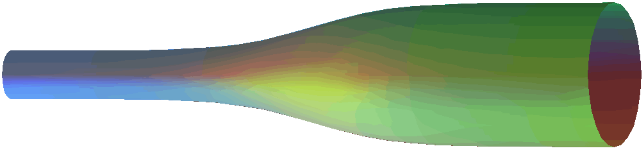

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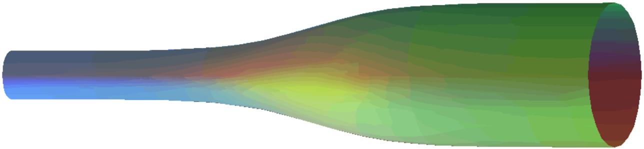

- Slow variation $\implies$ computationally intensive
- BUT slow variation $\implies$ amenable to analytical methods
  - (practical limitations : size of tunnel and resistance of the structure)

Electromagnetic 2-form

- Electromagnetic 2-form F is related to $\mathbf{E}$ and $\mathbf{B}$ as follows :

$$F = dt \wedge (E_x dx + E_y dy + E_z dz) - B_x\, dy \wedge dz - B_y\, dz \wedge dx - B_z\, dx \wedge dy \qquad (1)$$

- Perfect conductor boundary condition at $\mathcal{S} = 0$

$$d\mathcal{S} \wedge F = 0 \,\Big|_{\mathcal{S}=0} \qquad (2)$$

is equivalent to $\mathbf{n} \times \mathbf{E} = 0$ and $\mathbf{n} \cdot \mathbf{B} = 0$ where $\mathbf{n}$ is normal to $\mathcal{S}$

Maxwell's equations

- Maxwell's equations

$$\mathrm{d}F = 0, \qquad \varepsilon_0 \mathrm{d} \star F = -\varrho \star \widetilde{V} \tag{3}$$

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- for an unperturbed beam

$$V = \frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} \tag{4}$$

assuming that it is travelling close to the speed of light c

Potential decomposition

- Exploit cylindrical symmetry of waveguide :

$$F = \left(\partial_u \mathcal{H}^\Phi + \partial_u \mathcal{H}^B + \partial_\zeta \mathcal{H}^B - 2\partial^2_{u\zeta} W - \partial^2_{\zeta\zeta} W\right) \mathrm{d}\zeta \wedge \mathrm{d}u$$
$$+ \mathrm{d}u \wedge \left[\mathrm{d}_\perp \left(\partial_u W + \partial_\zeta W - \mathcal{H}^B\right) + \partial_u \#_\perp \mathrm{d}_\perp X\right]$$
$$+ \mathrm{d}\zeta \wedge \left[\mathrm{d}_\perp \left(\mathcal{H}^B - \partial_\zeta W\right) + \#_\perp \mathrm{d}_\perp (\partial_\zeta X - \mathcal{H}^\varphi)\right]$$
$$+ \left(\partial_\zeta \mathcal{H}^\varphi + \partial_u \mathcal{H}^\phi + \mathcal{H}^b - 2\partial^2_{u\zeta} X - \partial^2_{\zeta\zeta} X\right) \#_\perp 1 \qquad (5)$$

where $\zeta = z$, $u = z - ct$, $\mathrm{d}_\perp$ is the exterior derivative acting in the (x, y) subspace (transverse subspace) and $\#_\perp$ is the Hodge map on the transverse subspace

- Follows from Hodge decomposition $\Lambda_1(\mathcal{D}) = \mathrm{d}_\perp \mathcal{F}_d(\mathcal{D}) \oplus \#_\perp \mathrm{d}_\perp \mathcal{F}(\mathcal{D})$ on the transverse cross-section $\mathcal{D}$ of the waveguide

Field equations

Maxwell's equations may be reduced to :

$$\delta_\perp \mathrm{d}_\perp \mathcal{H}^B = 0, \tag{6}$$

$$\mathrm{d}_\perp \mathcal{H}^b = \#_\perp \mathrm{d}_\perp \left(\partial_u \mathcal{H}^B \right), \tag{7}$$

$$\mathrm{d}_\perp \mathcal{H}^\varphi = \#_\perp \mathrm{d}_\perp \mathcal{H}^\Phi, \tag{8}$$

$$\begin{aligned} \delta_\perp \mathrm{d}_\perp W - 2\partial^2_{u\zeta} W - \partial^2_{\zeta\zeta} W \\ + \partial_u \mathcal{H}^\Phi + \partial_u \mathcal{H}^B + \partial_\zeta \mathcal{H}^B = P(r,\theta,u), \end{aligned} \tag{9}$$

$$\delta_\perp \mathrm{d}_\perp X - 2\partial^2_{u\zeta} X - \partial^2_{\zeta\zeta} X + \partial_\zeta \mathcal{H}^\varphi + \partial_u \mathcal{H}^\varphi + \mathcal{H}^b = 0 \tag{10}$$

where $\delta_\perp \mathrm{d}_\perp$ is the transverse Laplacian and

$$\partial_u P(r,\theta,u) = \rho(r,\theta,u) \tag{11}$$

Boundary conditions

Cylindrical symmetry with $\mathcal{S}$ as $r = R(\zeta)$:

$$W = 0|_{r=R(\zeta)}, \tag{12}$$

$$\partial_r X = 0|_{r=R(\zeta)}, \tag{13}$$

$$\mathcal{H}^B = \partial_\zeta W|_{r=R(\zeta)}, \tag{14}$$

$$\mathcal{H}^\Phi = -R'(\zeta)\frac{1}{r}\partial_\theta X|_{r=R(\zeta)} \tag{15}$$

Asymptotic approximation scheme

- Introduce "slow variation" in ζ :

$$\check{R}(\epsilon\zeta) = R(\zeta) \tag{16}$$

where $1 \gg \epsilon > 0$

- Introduce "slow coordinate" $s = \epsilon\zeta$ and

$$\check{\chi} = \sum_{n=0}^{\infty} \epsilon^n \check{\chi}_n, \tag{17}$$

where

$$\check{\chi} \in \left\{ \check{W}, \check{X}, \check{\mathcal{H}}^B, \check{\mathcal{H}}^b, \check{\mathcal{H}}^\Phi, \check{\mathcal{H}}^\varphi \right\} \tag{18}$$

N.B. Assume potentials depend on ζ through s only.

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Yields a hierarchy of Poisson and Laplace equations

Longitudinal wake potential

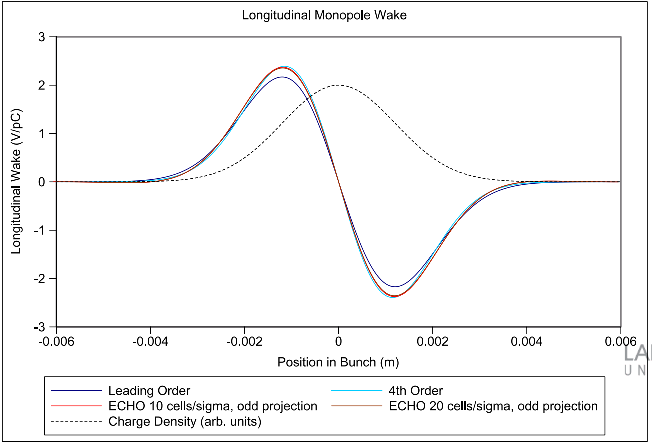
$$W^{\parallel}(r,\theta,u) = -\frac{1}{q_0}\int_{-\infty}^{\infty} E_z\Big|_{ct=z-u}\,dz \tag{19}$$

For simplicity, assume that the source is on-axis and gaussian in $u = z - ct$

Longitudinal wake potential : I

$$r = R(z) = 20 - 18\exp[-z^2/(8\times 10^5 l^2)], \qquad \epsilon = 1/\sqrt{8\times 10^5} \quad (20)$$

$l = 1$mm and z, r are measured in mm



Longitudinal wake potential : II

$$r = R(z) = 20 - 18\text{sech}(0.01z/l), \qquad \epsilon = 0.01 \qquad (21)$$

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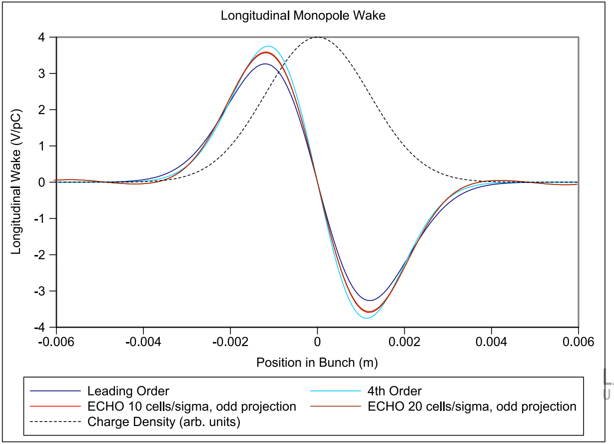

Impedance : on-axis harmonic charge density

$$Z^{\|}_{\text{on-axis}}(\omega) = Z^{\|}_{1\,\text{on-axis}} + Z^{\|}_{2\,\text{on-axis}} + Z^{\|}_{4\,\text{on-axis}} + Z^{\|}_{6\,\text{on-axis}} + \ldots \tag{22}$$

where

$$Z^{\|}_{1\,\text{on-axis}} = \frac{1}{2\pi\varepsilon_0 c} \ln\left(\frac{R_1}{R_2}\right), \tag{23}$$

$$Z^{\|}_{2\,\text{on-axis}} = -\frac{i\omega}{4\pi\varepsilon_0 c^2} \int_{-\infty}^{\infty} R'^2 \mathrm{d}z \tag{24}$$

$$Z^{\|}_{4\,\text{on-axis}} = \frac{i\omega}{96\pi\varepsilon_0 c} \int_{-\infty}^{\infty} \left\{ 5R'^4 + 3\left(R\,R''\right)^2 - 2\frac{\omega^2}{c^2}\left(R^2\,R''\right)^2 \right\} \mathrm{d}z \tag{25}$$

Impedance : on-axis harmonic charge density

$$
\begin{aligned}
&4\pi\varepsilon_0 c Z^{\|}_{6\,\text{on-axis}} = \\
&-\frac{i\omega}{c}\int_{-\infty}^{\infty}\left\{\frac{3}{16}(R''\,R'\,R)^2+\frac{11}{120}R'^6+\frac{1}{48}(R^2\,R''')^2\right\}\mathrm{d}z \\
&+\frac{i\omega^3}{c^3}\int_{-\infty}^{\infty}\left\{\frac{11}{256}\left(R^3\,R'''\right)^2-\frac{1}{6}\left(R^2\,R'\,R''\right)^2-\frac{73}{768}R^5\,R''^3\right\}\mathrm{d}z \\
&+\frac{i\omega^5}{c^5}\int_{-\infty}^{\infty}\left\{\frac{19}{160}(R^3\,R'\,R'')^2+\frac{73}{1920}R^7\,R''^3-\frac{19}{1920}(R'''\,R^4)^2\right\}\mathrm{d}z
\end{aligned}
\tag{26}
$$

- Similar integrals obtained for $Z^{\perp}(\omega)$

Optimal geometry

Stationary variations of an impedance

$$Z[R] = \int_{z_1}^{z_2} \lambda(R, R', R'', R''')\,\mathrm{d}z \qquad (27)$$

yield a BVP for an optimal geometry.

$$\begin{aligned} 0 = \delta Z = &\int_{z_1}^{z_2} \left(\frac{\partial \lambda}{\partial R} - \frac{d}{dz}\frac{\partial \lambda}{\partial R'} + \frac{d^2 \partial \lambda}{dz^2 \partial R''} - \frac{d^3 \partial \lambda}{dz^3 \partial R'''} \right) \delta R\, dz \\ &+ \left(\frac{\partial \lambda}{\partial R''} - \frac{d}{dz}\frac{\partial \lambda}{\partial R'''} \right) \delta R' \Big|_{z_1}^{z_2} \\ &+ \frac{\partial \lambda}{\partial R'''} \delta R'' \Big|_{z_1}^{z_2} \end{aligned} \qquad (28)$$

Optimal geometry

Hence, solve the ODE

$$\frac{\partial\lambda}{\partial R} - \frac{d}{dz}\frac{\partial\lambda}{\partial R'} + \frac{d^2\partial\lambda}{dz^2\partial R''} - \frac{d^3\partial\lambda}{dz^3\partial R'''} = 0, \tag{29}$$

subject to the natural BCs

$$\left(\frac{\partial\lambda}{\partial R''} - \frac{d}{dz}\frac{\partial\lambda}{\partial R'''}\right)\bigg|_{z_1,z_2} = 0, \tag{30}$$

$$\frac{\partial\lambda}{\partial R'''}\bigg|_{z_1,z_2} = 0 \tag{31}$$

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- 2nd order unaffected by 4th order and 6th order corrections
- Does this hold to all orders?
- (optimal geometry w.r.t. transverse impedance is frequency dependent)

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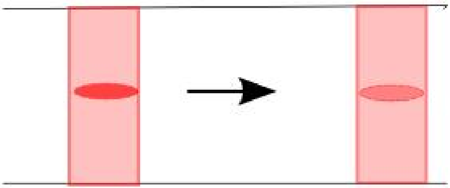

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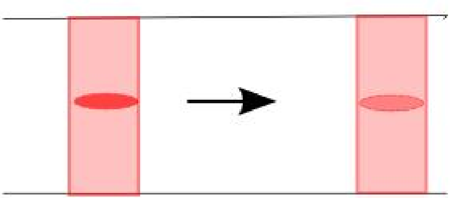

  - But sharp corners lead to wave emission, propagation and diffraction
    - $\Longrightarrow$ modal analysis

Reference

- “Wake potentials and impedances of charged beams in gradually tapering structures”
  DAB, DC Christie, JDA Smith, RW Tucker.
  arXiv: 0906.0948 [physics.acc-ph]
  (submitted)

We thank the Cockcroft Institute for support.