Gramians based model reduction for hybrid switched systems

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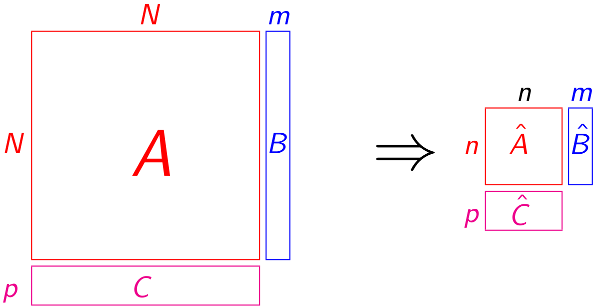
   - Stability analysis under restricted switching
4. Summary

Time invariant model reduction idea



$$\hat{A} = W^T A V,\ \hat{B} = W^T B,\ \hat{C} = CV \qquad \text{where } n \ll N$$

$$P = VW^T \text{ is a projector for } W^T V = I_n$$

$$y - \hat{y} = T_e(z)u := \left(\underbrace{C(zI_N - A)^{-1}B}_{T(z)} - \underbrace{\hat{C}(zI_n - \hat{A})^{-1}\hat{B}}_{\hat{T}(z)} \right) u$$

Balanced Reduction (Moore 81)

Lyapunov/Stein equations for system Gramians

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0 \qquad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0$$

$$A\mathcal{P}A^T + BB^T = \mathcal{P} \qquad A^T\mathcal{Q}A + C^TC = \mathcal{Q}$$

With $\mathcal{P} = \mathcal{Q} = \Sigma$: Want Gramians Diagonal and Equal

States Difficult to Reach are also Difficult to Observe

Reduced Model $\hat{A} = W_n^T A V_n$, $\hat{B} = W_n^T B$, $\hat{C} = C V_n$

$$\mathcal{P}V_n = W_n\Sigma_n, \qquad \mathcal{Q}W_n = V_n\Sigma_n$$

$$\mathcal{P} = \int_0^\infty x(\tau)x(\tau)^T d\tau = \int_0^\infty e^{A\tau}BB^Te^{A^T\tau}d\tau$$

$$\mathcal{Q} = \int_0^\infty e^{A^T\tau}C^TCe^{A\tau}d\tau$$

Hankel norm error estimate (Glover 84)

Why Balanced Trancation?

- Hankel singular values $= \sqrt{\lambda(\mathcal{PQ})}$
- Model reduction H_∞ error (Glover)

$$\|y - \hat{y}\| \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$

- Preserves Stability

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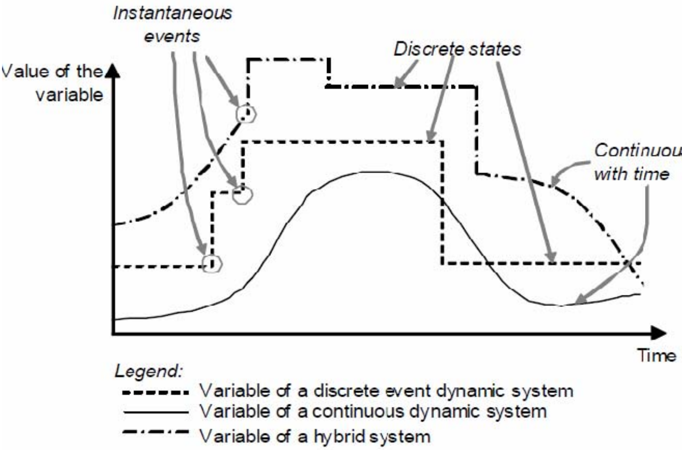
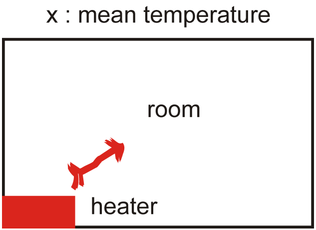
Switched dynamical systems

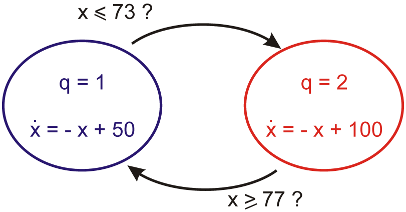
Hybrid system: Dynamical system that combines continuous with discrete-valued variables.

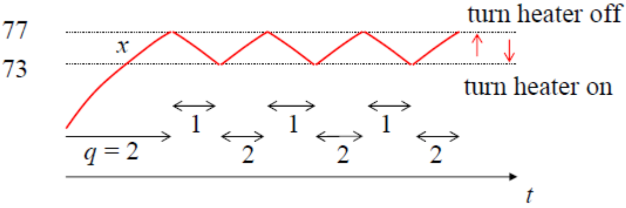


Behaviour of state variables of different nature

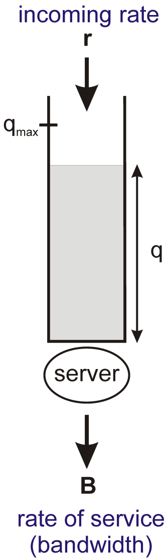
Example 1: Thermostat

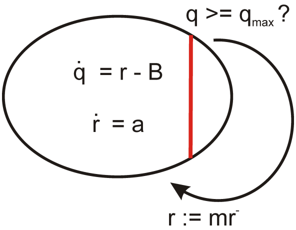


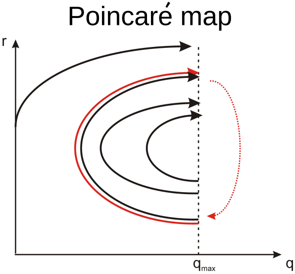


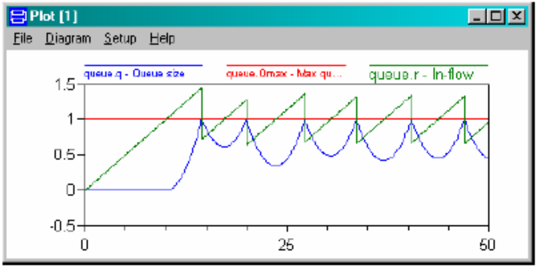


Example 2: Server system with congestion control









To think about ...

- For a given initial condition we can have many solutions.
- With hybrid systems **there are many possible notions of stability**.
  Which one is the best?

  (engineering question, not a mathematical one)

  What type of perturbations do you want to consider on the initial conditions?

  (this will define the topology on the initial conditions)

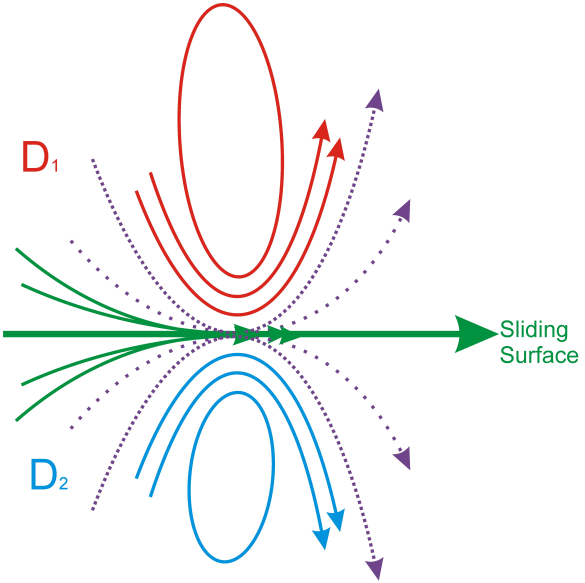
  What type of changes are you willing to accept in the solution?

  (this will define the topology on the signals)

- Even for the simple case:

  Lyapunov, integral, Poincare

  (even for linear systems these definitions may differ: Why?)

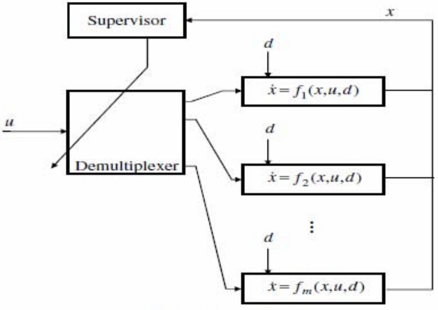

D_1
D_2
Sliding
Surface

Switched dynamical system II

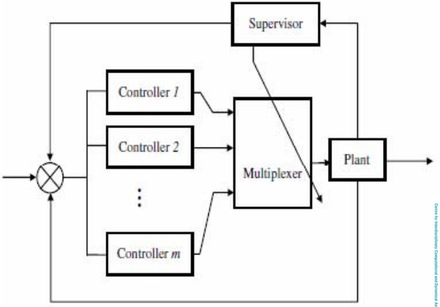
Continuous-time

$$\frac{d}{dt}x(t) = f_i\left(x(t), u(t)\right), \quad t \in \mathbb{R}^+, \quad i \in \mathcal{I} = \{1, \ldots, N\}$$

where: state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^m$, $\mathcal{I}$ an index set.



Schematic of a switched system



Schematic of a system with multiple controllers

Switching logic

- A logical rule orchestrating the switching between the subsystems.
- Usually described as classes of piecewise constant maps $\sigma : \mathbb{R}^+ \to \mathcal{I}$
- $\hookrightarrow$ $\sigma(t)$ has finite number discontinuities on any part of $\mathbb{R}^+$
- $\hookrightarrow$ No-chattering requirement, the sliding-like motion can be easily identified before hand.
- $i = \sigma(t)$ the active mode at time t

In general, the switching logic is time-controlled, state-dependent, and with memory

$$i = \sigma(t, x(t), \sigma(\tau)) = \sigma(t+), \quad \tau < t$$

Two cases

- time-transmission switching $\quad i = \sigma(t) = \sigma(t+), \quad \tau < t$
- time-invariant switching $\quad i = \sigma(x(t), \sigma(\tau)) = \sigma(t+), \quad \tau < t$

Motivations

- Many systems cannot be asymptotically stabilized by a single smooth feedback control law.
- Intelligent control by switching between different controllers to achieve stability and improve transient response.
- Most important basic problems:
  - Find conditions that guarantee the stability of the switched system for any switching signal.
  - Identify those classes of stabilizing switching signals.
  - Construct a stabilizing switching signal.
- Events detection is very important.
- All techniques are currently intractable.
- A reduced model will make feasible computational investigation. Especially: Formal verification

Adequate choice of the measure of quality

Formal verification of hybrid systems

A typical safety verification problem consists of determining if any trajectory of (Σ) starting in a given set of initial conditions S enters a given set of unsafe states B.

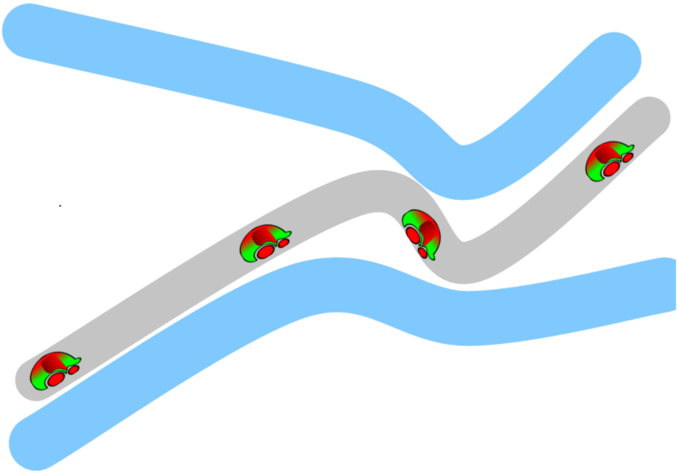
- Reduction based on an upper bound are not suitable

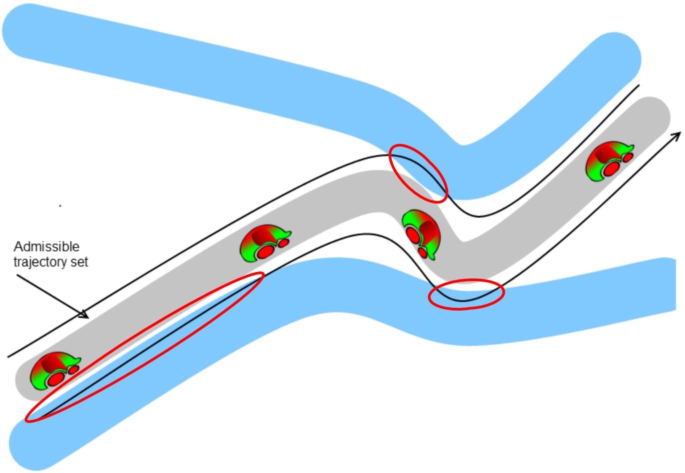
$$\|y(t) - \hat{y}(t)\| = \left(\int_0^\infty |y(t) - \hat{y}(t)|^2 dt \right)^{\frac{1}{2}}$$

- Instead use

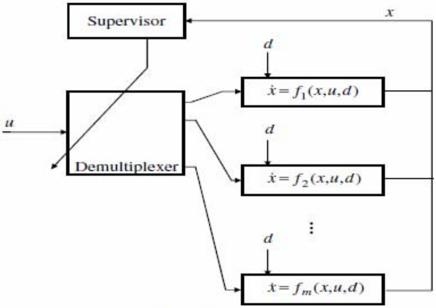
$$\|y(t) - \hat{y}(t)\| = \sup_{t \in [0,\infty)} |y(t) - \hat{y}(t)|$$

- Balanced truncation like method could be considered.

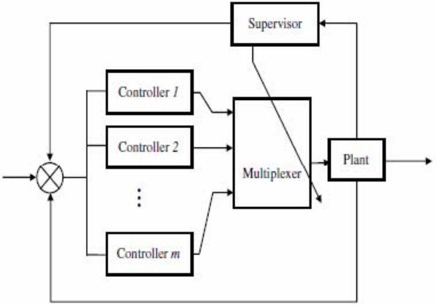



Admissible
trajectory set

Model reduction of switched linear systems (MoRSLS)



Schematic of a switched system



Schematic of a system with multiple controllers

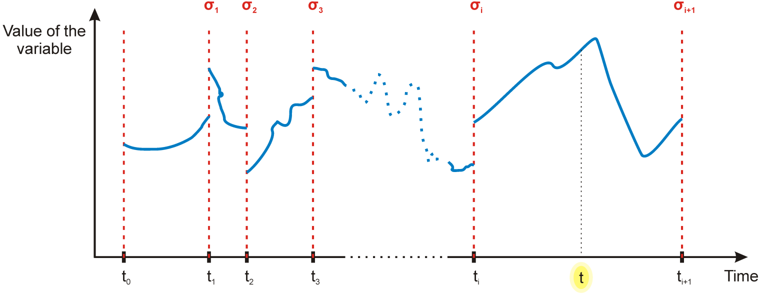
Hypothesis

- Each subsystem is considered to be asymptotically stable, controllable and observable.
- No restriction on the switching signals: arbitrary switching.

Different model order reduction scenarios

Scenario 1

Switching path is known a priori $\{(t_1, \sigma_1), (t_2, \sigma_2), \ldots, (t_N, \sigma_N)\}$



Use BT with the Gramians

$$\mathcal{P} = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{A_i \tau} B_i B_i^T e^{A_i^T \tau} d\tau$$

$$\mathcal{Q} = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{A_i^T \tau} C_i^T C_i e^{A_i \tau} d\tau$$

Scenario 1

Use BT with the Gramians

$$\mathcal{P} = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{A_i\tau} B_i B_i^T e^{A_i^T\tau} d\tau = \sum_{i=1}^{N} \alpha_i \mathcal{P}_i \qquad \sum_{i=1}^{N} \alpha_i = 1$$

$$\mathcal{Q} = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{A_i^T\tau} C_i^T C_i e^{A_i\tau} d\tau = \sum_{i=1}^{N} \beta_i \mathcal{Q}_i \qquad \sum_{i=1}^{N} \beta_i = 1$$

$$\mathcal{P}_i = \int_{t_{i-1}}^{t_i} e^{A_i\tau} B_i B_i^T e^{A_i^T\tau} d\tau \quad \text{or} \quad = \int_{t_0}^{t_N} e^{A_i\tau} B_i B_i^T e^{A_i^T\tau} d\tau$$

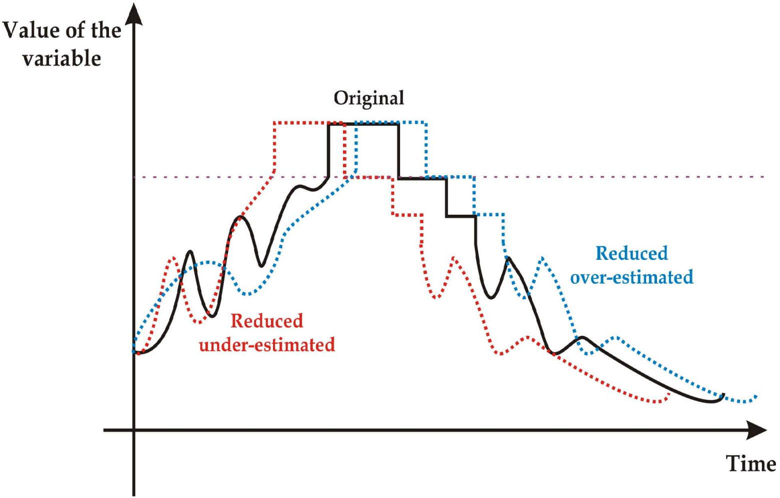
$$\text{or} \quad = \int_{0}^{\infty} e^{A_i\tau} B_i B_i^T e^{A_i^T\tau} d\tau$$

$$\mathcal{Q}_i = \int_{t_{i-1}}^{t_i} e^{A_i^T\tau} C_i^T C_i e^{A_i\tau} d\tau \quad \text{or} \quad = \int_{t_0}^{t_N} e^{A_i^T\tau} C_i^T C_i e^{A_i\tau} d\tau$$

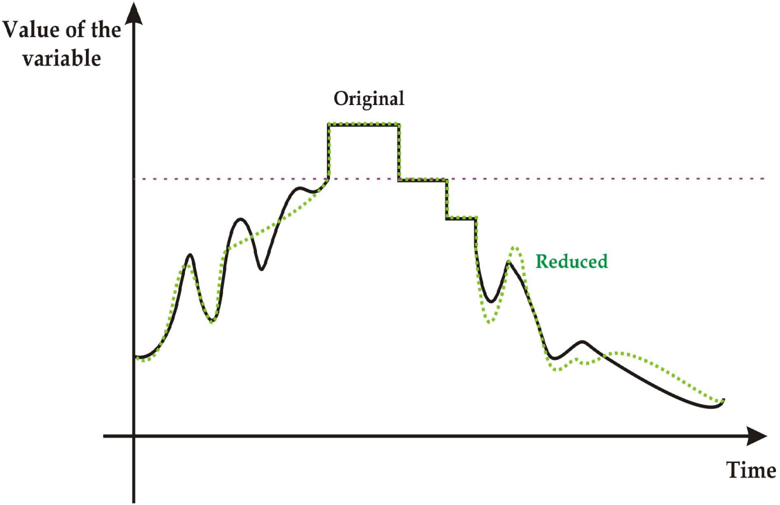
$$\text{or} \quad = \int_{0}^{\infty} e^{A_i^T\tau} C_i^T C_i e^{A_i\tau} d\tau$$

MoRSLS: scenario 2

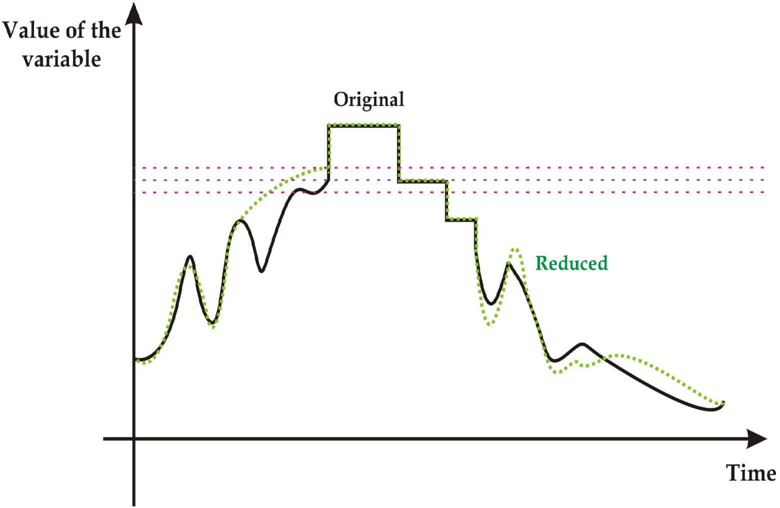
Reduce each subsystem independently and keep the same switching rule



MoRSLS: scenario 2a



MoRSLS: scenario 2b



MoRSLS: scenario 3 - Stability under arbitrary switching

The main question

Whether the switched system is stable when there is no restriction on the switching signals.

Assumptions

- It is necessary to require that all the subsystems are asymptotically stable.
- Also sufficient if:
  - A_i being pairwise commutative ($A_i A_j = A_j A_i$ for all $i, j \in \mathcal{I}$) (Narendra & all (1994) and Zhai & all (2002))
  - A_i symmetric ($A_i^T = A_i$ for all $i \in \mathcal{I}$) (Zhai & all (2004))
  - A_i normal ($A_i^T A_i = A_i A_i^T$ for all $i \in \mathcal{I}$) (Zhai & all (2006))
- The stability of the switched system is guaranteed under arbitrary switching if there exists a common Lyapunov function for all the subsystems.
- One has to solve a very large LMI. $PA_i + A_i^T P < 0, \quad \forall i \in \mathcal{I}$

Simple case:

two modes A_1 and A_2 in $\mathbb{R}^{2\times 2}$.

Shorten and Narendra (99, 03)

A necessary and sufficient condition for the existence of a common quadratic Lyapunov function can be based on the stability of the matrix pencil formed by the pair of subsystems state matrices:

$$\gamma_\alpha(A_1, A_2) = \alpha A_1 + (1-\alpha)A_2, \quad \alpha \in [0, 1].$$

$\gamma_\alpha(A_1, A_2)$ is said to be Hurwitz if

$$Real\,(\lambda\,(\gamma_\alpha(A_1, A_2))) < 0, \quad \text{for all} \quad 0 \le \alpha \le 1.$$

Theorem

Let A_1, A_2 be two Hurwitz matrices in $\mathbb{R}^{2\times 2}$. The following conditions are equivalent:

1. there exist a CQLF with A_1, A_2 as the two subsystems;
2. the matrix pencils $\gamma_\alpha(A_1, A_2)$ and $\gamma_\alpha(A_1, A_2^{-1})$ are Hurwitz;
3. the matrices $A_1 A_2$ and $A_1 A_2^{-1}$ do not have any negative real eigenvalues.

This result is difficult to generalize to higher dimensional systems.

Let A_1, A_2 be two Hurwitz matrices in $\mathbb{R}^{n \times n}$.

Theorem

A necessary condition for the existence of a CQLF is that the matrix products $A_1[\alpha A_1 + (1-\alpha)A_2]$ and $A_1[\alpha A_1 + (1-\alpha)A_2]^{-1}$ do not have any negative real eigenvalues for all $0 \leq \alpha \leq 1$.

Theorem

If $\mathrm{rank}(A_2 - A_1) = 1$. A necessary and sufficient condition for the existence of a CQLF for the switched system with A_1, A_2 as the two subsystems is that the matrix product $A_1 A_2$ does not have any negative real eigenvalues. Equivalently, the matrix $A_1 + \gamma A_2$ is non-singular for all $\gamma \in [0, +\infty)$.

Theorem Shorten and Narendra (2002)

Let $A_1, A_2, \ldots, A_N$ be a finite number of Hurwitz matrices in $\mathbb{R}^{2\times 2}$ with $a_{21i} \neq 0$ for all i. A necessary and sufficient condition for the existence of a CQLF is that a CQLF exists for every 3-tuple of systems $\{A_i, A_j, A_k\}$, $i \neq j \neq k$, for all $i, j, k \in \{1, \ldots, N\}$.

Common Lyaponuv function problems

- The standard interior point methods for LMIs may become ineffective as the number of modes increases.
- Liberzon & all (2004) proposed an interactive gradient decent algorithm, which could converge to the CQLF in finite number of steps. But the convergence is in the sense of probability one, and it is a numerical method.
- Algebraic conditions remain challenging task (Liberzon (2003) and Liberzon & all (1999)).
- Complex matrix pencils conditions
- Converse Lyaponuv theorems
- Recently, in Laffey & Smigoc (2007), A tensor condition was introduced as a necessary condition for the existence of a CQLF for the general case, i.e., a switched system consisting of a finite number of n-th order LTI systems.
- Alternatively, Liberzon & al. (1999) proposed a Lie algebraic condition for switched LTI systems, which is based on the solvability of the Lie algebra generated by the subsystems state matrices.

Other algorithms under investigation

The system

$$\dot{x} = A_i x, \quad i = 1, \ldots, N$$

has a common Lyaponuv function

$$V(x) := x^T \mathcal{P}_N x$$

where

$$\begin{aligned} A_1^T \mathcal{P}_1 + \mathcal{P}_1 A_1 &= -I, \\ A_i^T \mathcal{P}_i + \mathcal{P}_i A_i &= -\mathcal{P}_{i-1}, \quad i = 2, \ldots, N. \end{aligned}$$

$\mathcal{P}_i$ is the unique symmetric positive definite matrix solution of the corresponding equation.

Other algorithms under investigation

The system

$$\dot{x} = A_i x + B_i u, \quad i = 1, \ldots, N$$

has a common Lyaponuv function

$$V(x) := x^T \mathcal{P}_N x$$

where

$$\begin{aligned} A_1^T \mathcal{P}_1 + \mathcal{P}_1 A_1 &= -B_1 B_1^T \text{ or } \epsilon_{B_1}^2 I, \\ A_i^T \mathcal{P}_i + \mathcal{P}_i A_i &= -\epsilon_{B_i}^2 \mathcal{P}_{i-1}, \quad i = 2, \ldots, N. \end{aligned}$$

$\mathcal{P}_i$ is the unique symmetric positive definite matrix solution of the corresponding equation.

Other algorithms under investigation

The system

$$\dot{x} = A_i x + B_i u, \quad i = 1, \ldots, N$$

has a common Lyaponuv function

$$V(x) := x^T \mathcal{P}_N x$$

where

$$\begin{aligned} A_1^T \mathcal{P}_1 + \mathcal{P}_1 A_1 &= -\epsilon^2 I, \\ A_i^T \mathcal{P}_i + \mathcal{P}_i A_i &= -\epsilon^2 \mathcal{P}_{i-1}, \quad i = 2, \ldots, N. \end{aligned}$$

$$\epsilon = \epsilon_{B_1,\ \ldots,\ B_N}$$

$\mathcal{P}_i$ is the unique symmetric positive definite matrix solution of the corresponding equation.

Switched Quadratic Lyapunov Functions

(Daafouz & all. (2002) and Fang & all. (2004))

- A less conservative class of Lyapunov functions are the Switched Quadratic Lyapunov Functions:

  1. Every subsystem is stable, $\Rightarrow P_i = P_i^T \succ 0, \forall i \in \mathcal{I}$
  2. P_i are patched together based on the switching signals $\sigma(t)$ to construct a global Lyapunov function as

$$V(t, x(t)) = x^T(t) P_{\sigma(t)} x(t).$$

Theorem Fang & all. (2004)

If there exist positive definite symmetric matrices $P_i = P_i^T \in \mathbb{R}^{n\times n}$ and matrices $F_i, G_i \in \mathbb{R}^{n\times n}(i \in \mathcal{I})$, satisfying

$$\begin{bmatrix} A_iF_i^T + F_iA_i^T - P_i & A_iG_i - F_i \\ G_i^TA_i^T - F_i^T & P_j - G_i - G_i^T \end{bmatrix} \prec 0$$

for all $i, j \in \mathcal{I}$, then the switched discrete linear system is asymptotically stable.

The LMI can be replaced either by

$$\begin{bmatrix} P_i & A_i^TP_j \\ P_jA_i & P_j \end{bmatrix} \succ 0 \quad \text{or by} \quad \begin{bmatrix} -P_i & A_iG_i \\ G_i^TA_i^T & P_j - G_i - G_i^T \end{bmatrix} \prec 0$$

However, it is worth pointing out that the switched quadratic Lyapunov function method is still a sufficient only condition.

Necessary and Sufficient Stability Conditions

The asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-variant systems, for which several strong stability conditions exist.

$$x_{k+1} = A(k)x_k$$

where $A(k) \in \mathcal{A} = Conv\{A_1, A_2, \dots, A_N\}$.
Here, $Conv\cdot$ stands for convex combination.

Lemma (Bauer & all. 1993)

The linear time-variant system above is robustly asymptotically stable if and only if there exists a finite integer n such that

$$\|A_{i_1}A_{i_2}\dots A_{i_n}\|_\infty < 1$$

for all n-tuple $A_{i_j} \in \{A_1, A_2, \dots, A_N\}$, where $j = 1, \dots, n$.

Theorem (Lin & all. 2005)

A switched linear system $x_{k+1} = A_{\sigma(k)} x_k$, where $A_{\sigma(k)} \in \{A_1, A_2, \ldots, A_N\}$, is asymptotically stable under arbitrary switching if and only if there exists a finite integer n such that

$$\|A_{i_1} A_{i_2} \ldots A_{i_n}\|_\infty < 1$$

for all n-tuple $A_{i_j} \in \{A_1, A_2, \ldots, A_N\}$, where $j = 1, \ldots, n$.

Proposition

The following statements are equivalent:

1. The switched linear system $x_{k+1} = A_{\sigma(k)} x_k$, where $A_{\sigma(k)} \in \{A_1, A_2, \ldots, A_N\}$, is asymptotically stable under arbitrary switching;
2. the linear time-variant system $x_{k+1} = A(k) x_k$ where $A(k) \in \mathcal{A} = Conv\{A_1, A_2, \ldots, A_N\}$, is robustly asymptotically stable;
3. there exists a finite integer n such that

$$\|A_{i_1} A_{i_2} \ldots A_{i_n}\|_\infty < 1$$

for all n-tuple $A_{i_j} \in \{A_1, A_2, \ldots, A_N\}$, where $j = 1, \ldots, n$.

Stability analysis under restricted switching

- Slow switching
- Multiple Lyapunov Functions

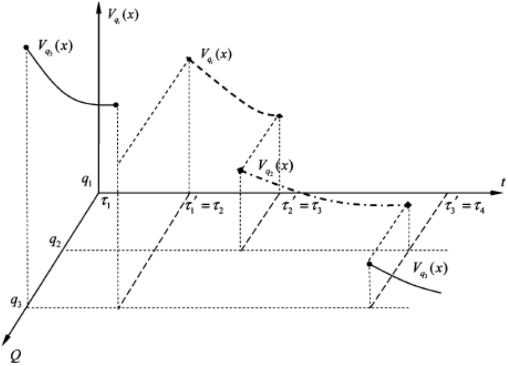


Fig. 1. The switched system is asymptotically stable if the Lyapunov-like functions' values at the switching instants form a decreasing sequence.

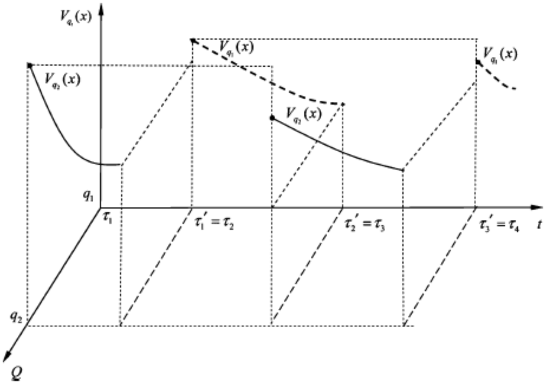


Fig. 2. For every subsystem, its Lyapunov-like function's value V_i at the start point of each interval exceeds the value at the start point of the next interval on which the i-th subsystem is activated, then the switched system is asymptotically stable.

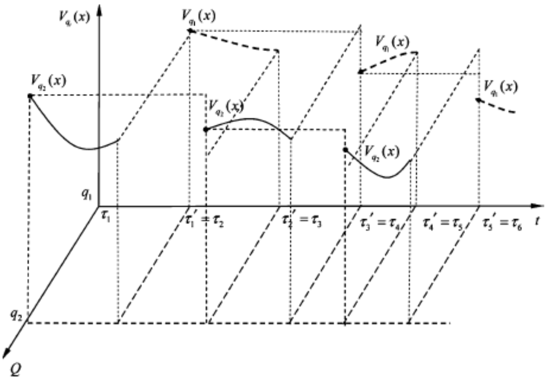


Fig. 3. The switched system can remain stable even when the Lyapunov-like function increases its value during certain period.

Piecewise Quadratic Lyapunov Functions (Pettersson & al. 2001)

A Lyapunov-like function $V_i(x) = x^T P_i x$ needs to satisfy the following two conditions

1. There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that

$$\alpha_i \|x\|^2 \leq V_i(x) \leq \beta_i \|x\|^2$$

hold for all $x \in \Omega_i$.
Consider a quadratic Lyapunov-like function candidate, and require that

$$\alpha_i x^T I x \leq x^T P_i x \leq \beta_i x^T I x$$

holds for any $x \in \Omega_i$. That is

$$\begin{cases} x^T(\alpha_i I - P_i)x \leq 0 \\ x^T(P_i - \beta_i I)x \leq 0 \end{cases}$$

holds for all $x \in \Omega_i$.

Piecewise Quadratic Lyapunov Functions (Pettersson & al. 2001)

2. For all $x \in \Omega_i$ and $x \neq 0$, $\dot{V}_i(x) < 0$.
   Equivalently, $\exists P_i = P_i^T$ such that

$$x^T[A_i^T P_i + P_i A_i]x < 0 \quad \text{for} \quad x \in \Omega_i.$$

3. Switching Condition:

$$x^T P_j x \leq x^T P_i x \quad \text{for} \quad x \in \Omega_{i,j} \subseteq \Omega_i \cap \Omega_j$$

S-procedure (Pettersson & al. 2002)

Let consider

$$\Omega_i = \{x | x^T Q_i x \geq 0\}, \quad \Omega_{i,j} = \{x | x^T Q_{i,j} x \geq 0\}$$

Theorem

The switched system is (exponentially) stable if there exist matrices $P_i = P_i^T$ and scalars $\alpha > 0$, $\beta > 0$, $\mu_i \geq 0$, $\nu_i \geq 0$, $\vartheta_i \geq 0$, and $\eta_{i,j} \geq 0$, such that

$$\begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A_i^T P_i + P_i A_i + \vartheta_i Q_i \leq -I \\ P_j + \eta_{i,j} Q_{i,j} \leq P_i \end{cases}$$

are satisfied.

References

- Lin H. & Antsaklis P.J., Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results. IEEE Trans. Automatic Control, vol. 54, no. 2, 2009.
- Sun, Z. & Ge, S. S. Switched linear systems. Control and design. Communications and Control Engineering. London: Springer., 2005.

Summary

- Switched dynamical systems
- Model reduction of switched dynamical systems
- Stability analysis under arbitrary switching
- Necessary and sufficient conditions for asymptotic stability
- A balanced truncation-like method for model reduction of switched dynamical systems
- Experiments are encouraging

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