Random Tri-Diagonal Operators

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...with the help of...

This talk is based on joint work with

- Marko Lindner, Chemnitz, Germany
- Brian Davies, KCL
- Ratchanikorn Chonchaiya, Reading

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What this talk is about and motivation

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Let U, V and W be compact sets in $\mathbb{C}$ and

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & v_{-1} & w_{-1} & & \\ & u_0 & v_0 & w_0 & \\ & & u_1 & v_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

with

$$u_i \in U,\ v_i \in V \text{ and } w_i \in W,$$

and let ...

Our Objects and Questions of Study

$$A_+ = \begin{pmatrix} v_1 & w_1 & & \\ u_2 & v_2 & w_2 & \\ & u_3 & v_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad A_n = \begin{pmatrix} v_1 & w_1 & & & \\ u_2 & v_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & w_{n-1} \\ & & & u_n & v_n \end{pmatrix}.$$

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Questions:

In the case that u_i, v_i, w_i are iid random variables:

- What are spectra of A, A_+, and A_n? How are they related?
- How to compute spectra of A and A_+?
- How to solve $Ax = b$ and $A_+x_+ = b_+$?

Motivation

Large interest in mathematical physics literature, e.g. **random Schrödinger operator**

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & v_{-1} & 1 & & \\ & 1 & v_0 & 1 & \\ & & 1 & v_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

with $v_i \in V \subset \mathbb{R}$.

Motivation

Large interest in mathematical physics literature,
e.g. **non-selfadjoint Anderson model** (Hatano & Nelson 1996)

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & v_{-1} & e^g & & \\ & e^{-g} & v_0 & e^g & \\ & & e^{-g} & v_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

with $g > 0$ and $v_i \in V = [-a, a]$.

Motivation

Large interest in mathematical physics literature,
e.g. **random hopping model** (Feinberg & Zee 1999)

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & \pm 1 & & \\ & \pm 1 & 0 & \pm 1 & \\ & & \pm 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

The picture is our best guess/computation of spec A.

MOPNet Connection?

- This is imperfect: standard spectra - not generalised versions - but this is hard enough!

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- This is imperfect: standard spectra - not generalised versions - but this is hard enough!
- But follow-on from:
  - Talk by **Marko Lindner** at MOPNet 0
  - Recent visit, part MOPNet-funded, to Reading by **Marko Lindner** and **Gil Strang**

We include here new results arising from that visit.

Spectral Set Definitions for Matrices and Operators

Matrix Spectrum

For a square matrix B, the **spectrum** is

$$\begin{aligned}\operatorname{spec} B &= \{\lambda \in \mathbb{C} : \lambda I - B \text{ not invertible}\} \\ &= \{\text{eigenvalues of } B\} = \{\text{eigenvalues of } B^T\}.\end{aligned}$$

Infinite Matrix Spectrum

For $B = A$ or A_+, the **spectrum** as an operator on $\ell^p = \{x : \|x\|_p < \infty\}$ is

$$\begin{aligned} \mathrm{spec}^p B &= \{\lambda \in \mathbb{C} : \lambda I - B \text{ not invertible as an operator on } \ell^p\} \\ &\supset \mathrm{spec}^p_{\mathrm{point}} B = \{ \text{ eigenvalues of } B \text{ (as operator on } \ell^p)\}. \end{aligned}$$

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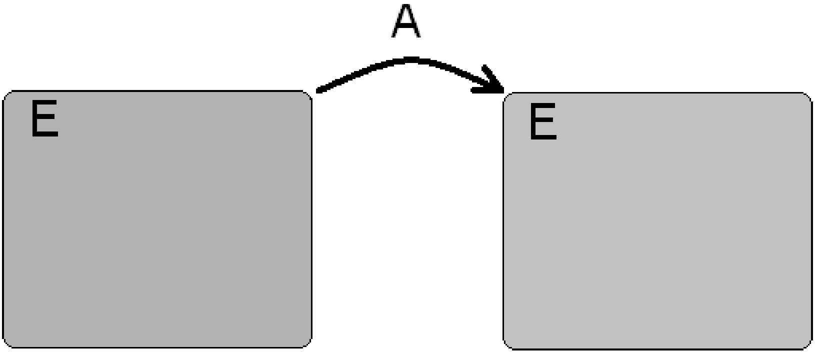
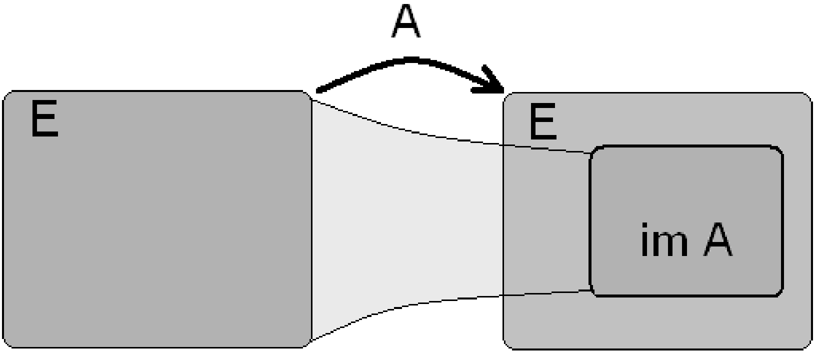
Helpfully, $\mathrm{spec}^p B$ is **same for all** p (so we abbreviate as $\mathrm{spec}\, B$).


A
E
E

Fredholm operators



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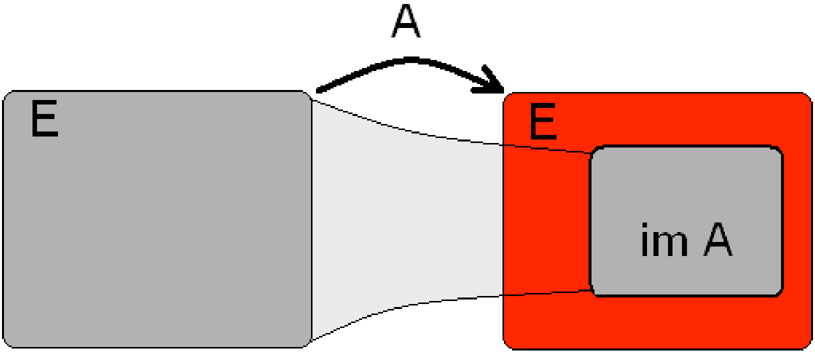

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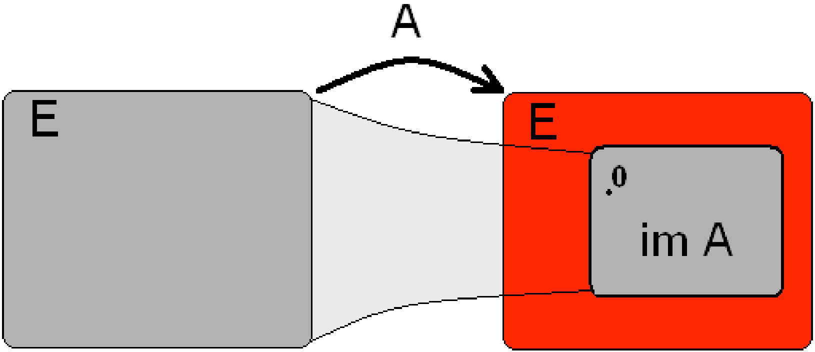

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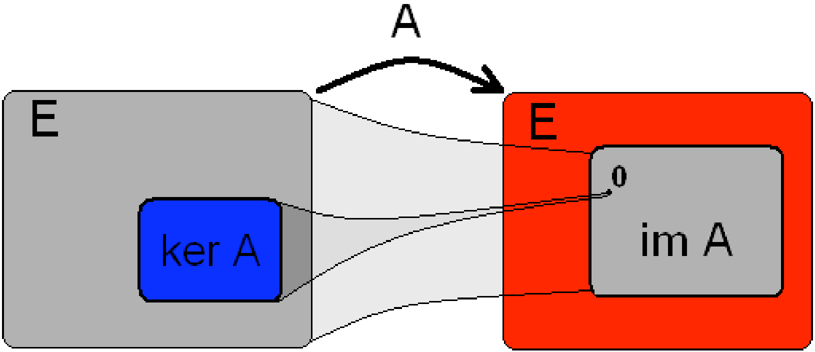

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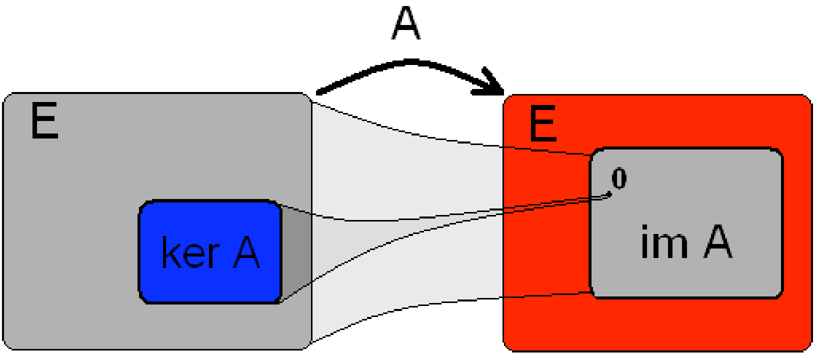
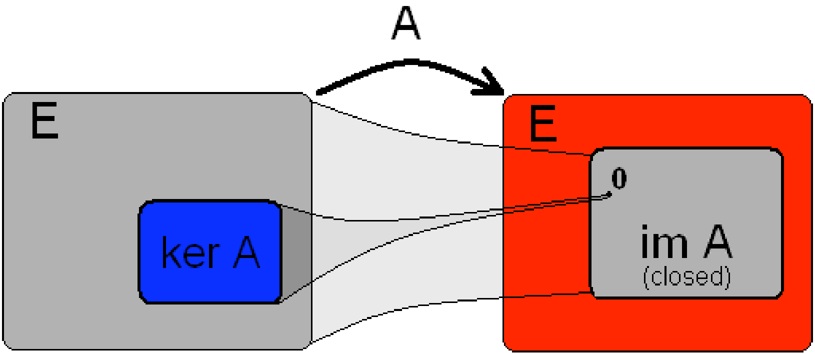


Definition

$A : E \to E$ is a *Fredholm operator*
if $\alpha := \dim(\ker A)$ and $\beta :=$ codim(im A) are both finite.
The difference $\alpha - \beta$ is then called the *index of A*.

Fredholm operators



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Infinite Matrix Essential Spectrum

For $B = A$ or A_+, the **essential spectrum** as an operator on $\ell^p = \{x : \|x\|_p < \infty\}$ is

$$\begin{aligned} \mathrm{spec}^p_{\mathrm{ess}} B &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ not Fredholm as an operator on } \ell^p\} \\ &\subset \mathrm{spec}\, B. \end{aligned}$$

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Helpfully (Lindner 2008), $\mathrm{spec}^p_{\mathrm{ess}} B$ is **same for all** p (so we abbreviate as $\mathrm{spec}_{\mathrm{ess}} B$).

An important distinction

- If $\lambda \in \mathrm{spec}\, B$ but $\notin \mathrm{spec}_{\mathrm{ess}} B$ then $B\phi = \lambda\phi$ or $B^T\phi = \lambda\phi$ with ϕ **exponentially localised**.

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- If $\lambda \in \mathrm{spec}\, B$ but $\notin \mathrm{spec}_{\mathrm{ess}} B$ then $B\phi = \lambda\phi$ or $B^T\phi = \lambda\phi$ with ϕ **exponentially localised**.
- If $\lambda \in \mathrm{spec}_{\mathrm{ess}} B$ then, for every $\varepsilon > 0$, $\|B\phi - \lambda\phi\|_\infty < \varepsilon$, with ϕ **not localised**, i.e. $|\phi_m| \not\to 0$ as $|m| \to \infty$.

Limit Operators: Definition

Limit operators: a key concept for understanding and characterising spectral sets for infinite matrices.

Definition

We call $B = (b_{ij})_{i,j\in\mathbb{Z}}$ a **limit operator** of $A = (a_{ij})_{i,j\in\mathbb{Z}}$ if there is a sequence $h_1, h_2, \ldots \in \mathbb{Z}$ with $|h_n| \to \infty$ such that, for all $i, j \in \mathbb{Z}$,

$$a_{i+h_n, j+h_n} \to b_{ij} \quad \text{as} \quad n \to \infty.$$

We write A_h instead of B, where $h = (h_1, h_2, \ldots)$.

Limit Operators: An Example

Example: Let A be a discrete Schrödinger operator

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & b_{-1} & 1 & & \\ & 1 & b_0 & 1 & \\ & & 1 & b_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

with the following potential

$$b = (\ldots, \underbrace{\beta, \beta, \beta, \beta}_{4}, \underbrace{\alpha, \alpha, \alpha}_{3}, \underbrace{\beta, \beta}_{2}, \underbrace{\alpha}_{1}, \underbrace{\beta, \beta}_{2}, \underbrace{\alpha, \alpha, \alpha}_{3}, \underbrace{\beta, \beta, \beta, \beta}_{4}, \ldots).$$

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Then all limit operators of A are of the form

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \beta & 1 & \\ & 1 & \beta & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha & 1 & \\ & 1 & \alpha & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

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or they are translates of the latter two matrices.

Example: Let again A be a discrete Schrödinger operator

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & b_{-1} & 1 & & \\ & 1 & b_0 & 1 & \\ & & 1 & b_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

but now with $\ldots, b_{-1}, b_0, b_1, \ldots$ independent **samples from a random variable** with values in a compact set $\Sigma \subset \mathbb{C}$.

Then, with probability 1, $b = (\ldots, b_{-1}, b_0, b_1, \ldots)$ is a **pseudo-ergodic** sequence over Σ, by which we mean the following:

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Definition (Davies 2001)

A sequence $b = (b_k)_{k\in\mathbb{Z}}$ is called **pseudo-ergodic** over $\Sigma \subset \mathbb{C}$ if every finite vector $c = (c_i)_{i\in F}$ with values $c_i \in \Sigma$ can be found, up to arbitrary precision $\varepsilon > 0$, somewhere inside the infinite sequence b, i.e.

$$\forall \varepsilon > 0 \quad \exists m \in \mathbb{Z} : \quad \max_{i\in F} |b_{i+m} - c_i| \; < \; \varepsilon.$$

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$$\forall\varepsilon > 0 \quad \exists m \in \mathbb{Z} : \quad \max_{i\in F} |b_{i+m} - c_i| \; < \; \varepsilon.$$

If the potential b is pseudo-ergodic then **every** discrete Schrödinger operator with a potential $c = (..., c_{-1}, c_0, c_1, ...)$ over Σ (including A itself) is a limit operator of A – and vice versa.

Limit Operators and Fredholmness

Let A be our bi-infinite tri-diagonal matrix.

Then it is not hard to see that

A **Fredholm** $\Longrightarrow$ all limit operators of A are **invertible**.

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Theorem Rabinovich, Roch, Silbermann 1998; Lindner 2003

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- all limit operators of A are **invertible** on ℓ^p,
- all limit operators of A are **injective** on ℓ^∞.

...moreover, the **Fredholm index** of A does **not** depend on p.

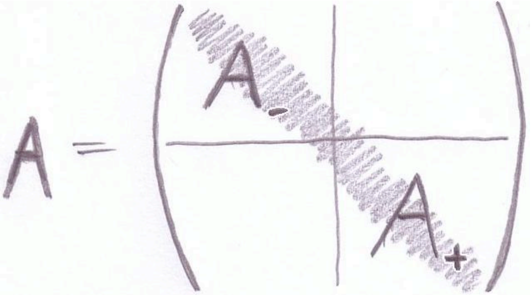
If we repeat the same argument with $\lambda I - A$ in place of A, we get:

Essential Spectrum Rabinovich, Roch, Silbermann 1998; Lindner 2003; C-W, Lindner 2008

$$\mathrm{spec}^p_{\mathrm{ess}}(A) = \bigcup_h \mathrm{spec}^p(A_h) = \bigcup_h \mathrm{spec}^\infty_{\mathrm{point}}(A_h), \qquad p \in [1, \infty]$$

Limit operators and the Fredholm index

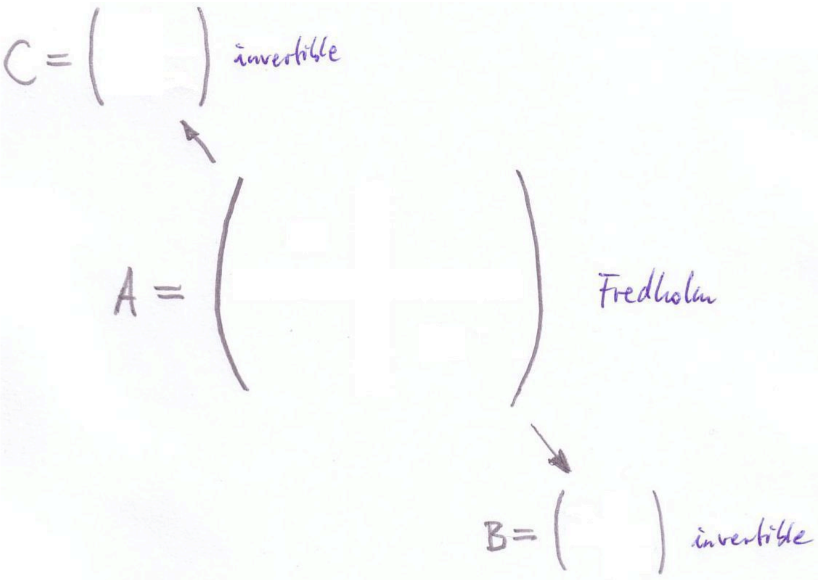
Think of our bi-infinite banded matrix A as a 2×2 block matrix:

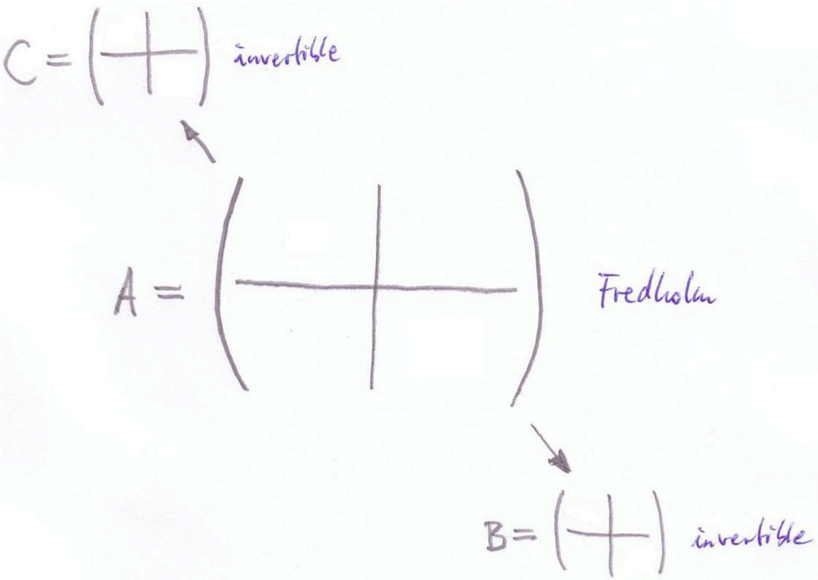


Then

$$
\begin{aligned}
A \text{ is Fredholm} \quad &\text{iff} \quad A_+ \text{ and } A_- \text{ are Fredholm}\\
\text{spec}_{\text{ess}} A \quad &= \quad \text{spec}_{\text{ess}} A_+ \ \cup \ \text{spec}_{\text{ess}} A_-\\
\text{ind}\, A \quad &= \quad \text{ind}\, A_+ \ + \ \text{ind}\, A_-
\end{aligned}
$$

Limit operators and the Fredholm index




C =
invertible
A =
Fredholm
B =
invertible

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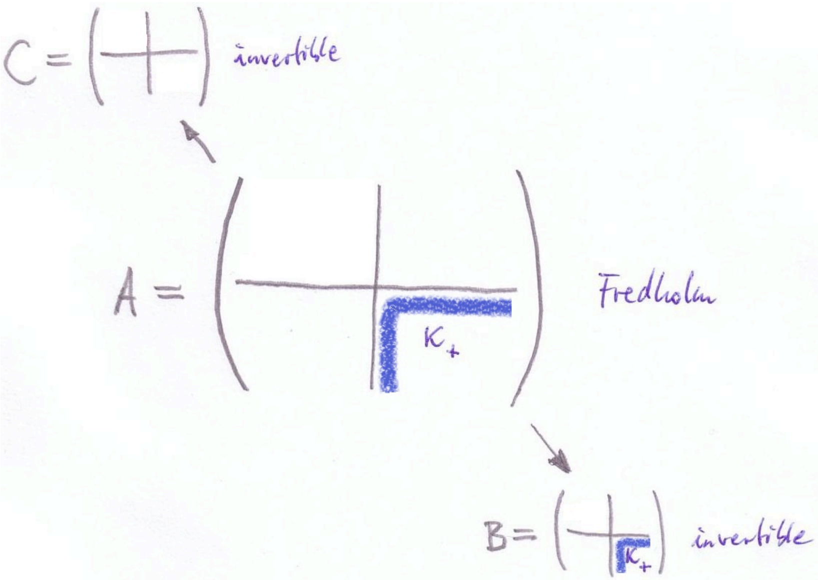

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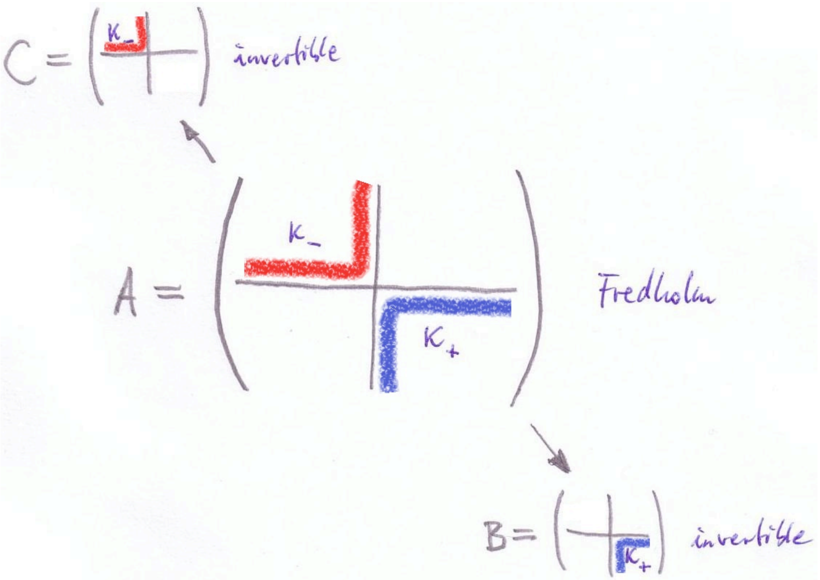
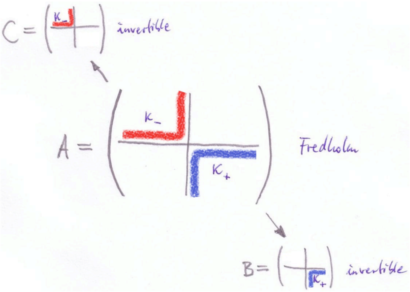


Limit operators and the Fredholm index

$$\text{ind}\, A \;=\; \text{ind}\, A_+ \;+\; \text{ind}\, A_-$$



Theorem Rabinovich, Roch, Roe 2004

$$\text{ind}\, A_+ \;=\; \text{ind}\, B_+ \quad \text{for all limops } B \text{ of } A \text{ at } +\infty$$
$$\text{ind}\, A_- \;=\; \text{ind}\, C_- \quad \text{for all limops } C \text{ of } A \text{ at } -\infty$$

Essential Spectrum

$$\mathrm{spec}_{\mathrm{ess}}\, A_+ \;=\; \bigcup_h \mathrm{spec}^{\infty}_{\mathrm{point}} A_h.$$

If A_+ is Fredholm then

Index

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Spectrum

$$\begin{aligned}\mathrm{spec}\, A_+ \quad = \quad & \mathrm{spec}_{\mathrm{ess}}\, A_+ \cup \{\lambda : \mathrm{ind}\,(\lambda I_+ - A_+) \neq 0\} \\ & \cup \{\lambda : \mathrm{ind}\,(\lambda I_+ - A_+) = 0 \text{ and } \alpha(\lambda I_+ - A_+) \neq 0\}.\end{aligned}$$

Limit Operator Methods Applied in the Random Case

The Random Case: Limit Operators

Let U, V and W be compact sets in $\mathbb{C}$ and

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-1} & & & & \\ & u_{-2} & v_{-1} & w_0 & & & \\ & & u_{-1} & v_0 & w_1 & & \\ & & & u_0 & v_1 & w_2 & \\ & & & & u_1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \qquad (1)$$

with random iid entries

$$u_i \in U, \; v_i \in V \text{ and } w_i \in W. \qquad (2)$$

Then, with probability one, A is **pseudo-ergodic**: the set of all limit operators of A equals the set of **all** operators of the form (1) with entries (2).

The Random Case: Limit Operators

Put

$$\begin{aligned} J(U,V,W) &:= \{ \text{ matrices (1) } : u_i \in U,\ v_i \in V,\ w_i \in W\}, \\ \Psi E(U,V,W) &:= \{ A \in J(U,V,W) : A \text{ pseudo-ergodic } \}. \end{aligned}$$

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-1} & & & & \\ & u_{-2} & v_{-1} & w_0 & & & \\ & & u_{-1} & v_0 & w_1 & & \\ & & & u_0 & v_1 & w_2 & \\ & & & & u_1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \qquad (1)$$

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For $A \in \Psi E(U,V,W)$, the set of limit operators is **all of** $J(U,V,W)$.

The Random Case: Spectral Sets

Put

$$\begin{aligned} J(U,V,W) &:= \{\text{ matrices (1) } : u_i \in U,\ v_i \in V,\ w_i \in W\}, \\ \Psi E(U,V,W) &:= \{\ A \in J(U,V,W)\ :\ A \text{ pseudo-ergodic } \}. \end{aligned}$$

For $A \in \Psi E(U,V,W)$, the set of limit operators is **all of** $J(U,V,W)$. Hence, the following are equivalent:

- A is Fredholm,
- A is invertible,
- all $B \in J(U,V,W)$ are invertible.

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In other words, for $A \in \Psi E(U,V,W)$:

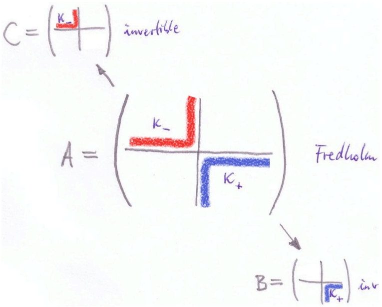
$$\operatorname{spec}_{\mathrm{ess}} A \;=\; \operatorname{spec} A \;=\; \bigcup \operatorname{spec} B,$$

with the union taken over all $B \in J(U,V,W)$.

The Random Case: Fredholm Index

If A is pseudo-ergodic and Fredholm, then, for **all** choices $u \in U$, $v \in V$ and $w \in W$:

$$\begin{aligned}
\operatorname{ind} A_- &= \operatorname{ind} \begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w \\ & u & v \end{pmatrix} \\
\operatorname{ind} A_+ &= \operatorname{ind} \begin{pmatrix} v & w & \\ u & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix}
\end{aligned}$$



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$$\operatorname{ind} A_- \;=\; \operatorname{ind}\begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & w \\ & u & v \end{pmatrix} \;=\; \operatorname{wind}(E_{u,w}, v)$$

$$\operatorname{ind} A_+ \;=\; \operatorname{ind}\begin{pmatrix} v & w & \\ u & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix} \;=\; -\operatorname{wind}(E_{u,w}, v)$$

$E_{u,w}$ is ellipse with half-axes of length $|u| + |w|$ and $|\,|u| - |w|\,|$, focal points $\pm 2\sqrt{uw}$.

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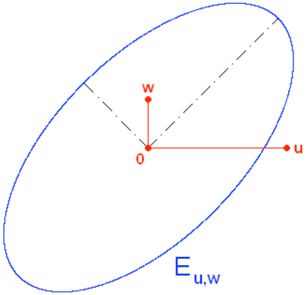

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The Random Case: Spectrum of A_+

For A_+ pseudo-ergodic:

$$\mathrm{spec}_{\mathrm{ess}} A_+ = \bigcup \mathrm{spec}\, B = \mathrm{spec}_{\mathrm{ess}} A = \mathrm{spec}\, A$$

with the union taken over all $B \in J(U, V, W)$, and

General Tri-Diagonal Formula Lindner & Roch 2011

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Special Case $U = \{u\}$, $V = \{v\}$, $W = \{w\}$

In this case A is a **Laurent operator**, and A_+ the associated (tri-diagonal) **Toeplitz operator**, and our results simplify to:

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If the Toeplitz operator A_+ is Fredholm, then $\alpha(A_+) = 0$ or $\beta(A_+) = 0$.

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Example C-W & Davies 2011

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & \pm\sigma & 0 & 1 & \\ & & \pm\sigma & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

Solving $A_+ x = b$

Obvious idea (**finite section method**) is to approximate

$$A_+ x_+ = b_+ \quad \text{by} \quad A_n x_n = b_n.$$

We hope that this is **stable**, i.e. $\sup_{n \geq N} \|A_n^{-1}\| < \infty$, and convergent, i.e. solution of $A_n x_n = b_n$, extended by zeros, converges to $A_+^{-1} b_+$.

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Theorem Baxter 1962, Gohberg & Feldman 1974

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Theorem — Lindner & Roch 2011, C-W & Lindner 2011

If A_+ is pseudo-ergodic, then the finite section method is stable, in fact

$$\sup_{n \geq 1} \|A_n^{-1}\| < \infty.$$

Conclusion

For the random tri-diagonal matrices A (bi-infinite) and A_+ (semi-infinite) we have, almost surely, that A and A_+ are pseudo-ergodic, which leads to:

- Characterisation of $\mathrm{spec}_{\mathrm{ess}} A_+ = \mathrm{spec}_{\mathrm{ess}} A = \mathrm{spec}\, A$ as union of ℓ^∞ eigenvalues of all possible operators with values in U, V, W.
- Explicit formula, as intersection of ellipses, for a set S such that $\mathrm{spec}\, A_+ = S \cup \mathrm{spec}_{\mathrm{ess}} A_+$, via an index formula and a Coburn's lemma.
- That if A_+ is invertible, then the finite section methods for approximating $A_+^{-1} b_+$ is stable.

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