Variational principles for eigenvalues of operator functions

Matthias Langer

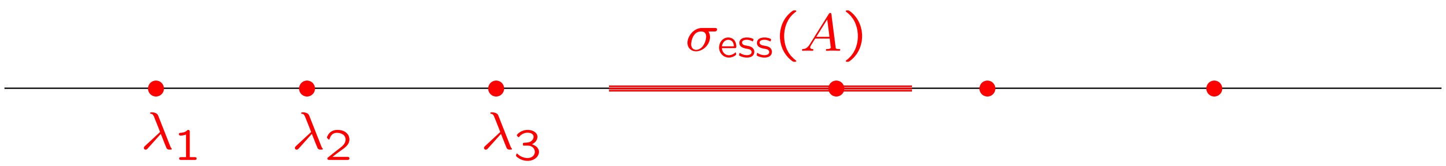
University of Strathclyde, Glasgow

Classical variational principle for eigenvalues

$\mathcal{H}$ a Hilbert space

A self-adjoint operator in $\mathcal{H}$, bounded from below,

i.e. $(Ax, x) \geq c\|x\|^2$ for all $x \in \mathrm{dom}(A)$ and some $c \in \mathbb{R}$.



For the eigenvalues λ_n below the essential spectrum:

$$\lambda_n = \min_{\substack{L \subset \mathrm{dom}(A) \\ \dim L = n}} \; \max_{\substack{x \in L \\ x \neq 0}} \frac{(Ax, x)}{\|x\|^2}$$

$$n = 1, 2, \ldots$$

$$\lambda_n = \max_{\substack{L \subset \mathcal{H} \\ \dim L = n-1}} \; \min_{\substack{x \perp L \\ x \in \mathrm{dom}(A),\, x \neq 0}} \frac{(Ax, x)}{\|x\|^2}$$

[H. Weber, Lord Rayleigh, H. Poincaré, E. Fischer, W. Ritz, . . .]

Essential spectrum

If A has N eigenvalues below the essential spectrum, then

$$\inf_{\substack{L\subset \mathrm{dom}(A)\\ \dim L=n}} \max_{\substack{x\in L\\ x\neq 0}} \frac{(Ax,x)}{\|x\|^2} = \min \sigma_{\mathrm{ess}}(A), \quad n > N.$$

Here $\lambda \in \sigma_{\mathrm{ess}}(A)$ if $A-\lambda$ is not Fredholm, i.e.

$\dim\ker(A-\lambda)=\infty$ or $\dim(\mathcal{H}/\operatorname{ran}(A-\lambda))=\infty$.

Comparison of two operators

If dom$(A) \supset$ dom(B) and

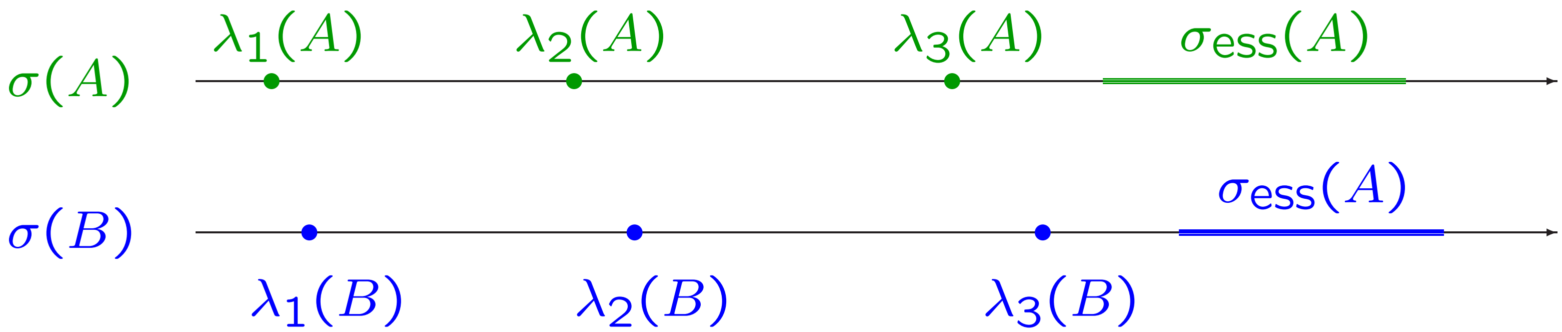
$$(Ax, x) \le (Bx, x), \qquad x \in \mathrm{dom}(B),$$

then

$$\lambda_n(A) \le \lambda_n(B)$$

and

$$\min \sigma_{\mathrm{ess}}(A) \le \min \sigma_{\mathrm{ess}}(B).$$



Operator functions and their spectra

$$T\colon \lambda \mapsto T(\lambda), \qquad \lambda \in \boldsymbol{\Delta}$$

where $T(\lambda)$ are closed operators in a Hilbert space $\mathcal{H}$ and $\boldsymbol{\Delta} \subset \mathbb{C}$.

Examples: $T(\lambda) = \lambda^2 A + \lambda B + C$, $\quad T(\lambda) = A - \lambda - B(D - \lambda)^{-1}C$

Operator functions and their spectra

$$T\colon \lambda \mapsto T(\lambda), \qquad \lambda \in \Delta$$

where $T(\lambda)$ are closed operators in a Hilbert space $\mathcal{H}$ and $\Delta \subset \mathbb{C}$.

Examples: $T(\lambda) = \lambda^2 A + \lambda B + C, \quad T(\lambda) = A - \lambda - B(D-\lambda)^{-1}C$

λ_0 eigenvalue of T $:\Longleftrightarrow$ $\exists\, x_0 \neq 0,$ such that $T(\lambda_0)x_0 = 0$

$\lambda_0 \in \sigma(T)$ (spectrum) $:\Longleftrightarrow$ $T(\lambda_0)$ not boundedly invertible
($\Longleftrightarrow 0 \in \sigma(T(\lambda_0))$)

$\lambda_0 \in \sigma_{ess}(T)$ (essential spectrum) $:\Longleftrightarrow$ $T(\lambda_0)$ is not Fredholm

NB. The usual eigenvalue problem $Ax = \lambda x$ for an operator A corresponds to the operator function

$$T(\lambda) = A - \lambda.$$

Generalised Rayleigh functional

Classical Rayleigh functional

$$\frac{(Ax, x)}{(x, x)} \qquad \text{for eigenvalue problem } Ax = \lambda x$$

is the zero of the function

$$\lambda \quad \mapsto \quad (Ax, x) - \lambda(x, x) = ((A - \lambda)x, x)$$

Generalised Rayleigh functional

Classical Rayleigh functional

$$\frac{(Ax,x)}{(x,x)} \qquad \text{for eigenvalue problem } Ax = \lambda x$$

is the zero of the function

$$\lambda \quad \mapsto \quad (Ax,x) - \lambda(x,x) = ((A-\lambda)x,x)$$

For the problem $T(\lambda)x = 0$ take a zero of

$$\lambda \quad \mapsto \quad (T(\lambda)x,x)$$

If λ_0 is an eigenvalue of T with eigenvector x_0,
then λ_0 is such a zero for $x = x_0$.

Generalised Rayleigh functional (continued)

Let T be defined on $\Delta \subset \mathbb{R}$ and assume that $T(\lambda)$ is self-adjoint if $\lambda \in \Delta$.

We assume that $(T(\cdot)x, x)$ is continuous and decreasing at value zero for $x \neq 0$, i.e. if $(T(\lambda_0)x, x) = 0$, then

$$(T(\lambda)x, x) > 0 \quad \text{for } \lambda < \lambda_0$$

$$(T(\lambda)x, x) < 0 \quad \text{for } \lambda > \lambda_0$$

Generalised Rayleigh functional (continued)

Let T be defined on $\Delta \subset \mathbb{R}$ and assume that $T(\lambda)$ is self-adjoint if $\lambda \in \Delta$.

We assume that $(T(\cdot)x, x)$ is continuous and decreasing at value zero for $x \neq 0$, i.e. if $(T(\lambda_0)x, x) = 0$, then

$$(T(\lambda)x, x) > 0 \quad \text{for } \lambda < \lambda_0$$
$$(T(\lambda)x, x) < 0 \quad \text{for } \lambda > \lambda_0$$

Generalised Rayleigh functional:

$$p(x) := \begin{cases} \lambda_0 & \text{if } (T(\lambda_0)x, x) = 0 \\ -\infty & \text{if } (T(\lambda)x, x) < 0 \quad \forall \lambda \in \Delta \\ \infty & \text{if } (T(\lambda)x, x) > 0 \quad \forall \lambda \in \Delta \end{cases}$$

Theorem. [Duffin, Rogers, Werner, Abramov, Markus, . . .]
Let T be an operator function defined on some interval $[\alpha, \beta)$ whose values are self-adjoint operators in $\mathcal{H}$. Assume that

(i) $\mathcal{D} = \operatorname{dom} T(\lambda)$ is independent of λ
(or $\operatorname{dom} T(\lambda) \subset \mathcal{D} \subset$ form domain of $T(\lambda)$);

(ii) T is continuous in norm resolvent sense;

(iii) $(T(\cdot)x, x)$ is decreasing at value 0 for all $x \in \mathcal{D}$;

(iv) $T(\alpha) > 0$.

Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of T below the essential spectrum, then

$$\lambda_n = \min_{\substack{L \subset \mathcal{D} \\ \operatorname{dim} L = n}} \max_{\substack{x \in L \\ x \neq 0}} p(x)$$

$$= \max_{\substack{L \subset \mathcal{H} \\ \operatorname{dim} L = n-1}} \min_{\substack{x \perp L \\ x \in \mathcal{D},\, x \neq 0}} p(x).$$

Comparison of two operator functions

Let T, $\widehat{T}$ be two operator functions as before such that $\mathcal{D} \supset \widehat{\mathcal{D}}$ and

$$p(x) \leq \widehat{p}(x), \qquad x \in \widehat{\mathcal{D}}. \tag{$*$}$$

Then

$$\lambda_n \leq \widehat{\lambda}_n.$$

Relation $(*)$ is satisfied if

$$(T(\lambda)x, x) \leq (\widehat{T}(\lambda)x, x) \qquad \text{for all } \lambda \in [\alpha, \beta), \quad x \in \widehat{\mathcal{D}}.$$

Theorem. [Binding, Eschwé, H. Langer 2000; Eschwé, M.L. 2004] Let T be an operator function defined on some interval $[\alpha, \beta)$ whose values are self-adjoint operators in $\mathcal{H}$. Assume that

(i) $\mathcal{D} = \mathrm{dom}(T(\lambda))$ is independent of λ
(or $\mathrm{dom}\, T(\lambda) \subset \mathcal{D} \subset$ form domain of $T(\lambda)$);

(ii) T is continuous in norm resolvent sense;

(iii) $(T(\cdot)x, x)$ is decreasing at value 0 for all $x \in \mathcal{D}$;

(iv) negative spectrum of $T(\alpha)$ consists of κ eigenvalues.

Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of T below the essential spectrum, then

$$\lambda_n = \min_{\substack{L \subset \mathcal{D} \\ \dim L = n+\kappa}} \max_{\substack{x \in L \\ x \neq 0}} p(x)$$

$$= \max_{\substack{L \subset \mathcal{H} \\ \dim L = n-1+\kappa}} \min_{\substack{x \perp L \\ x \in \mathcal{D},\, x \neq 0}} p(x).$$

Theorem (continued).

Let T be differentiable and assume in addition that the Virozub–Matsaev (VM) condition is satisfied:

$\exists\, \varepsilon, \delta > 0$ so that for all $x \in \mathcal{D}$, $\|x\| = 1$, and $\lambda \in (\alpha, \beta)$,

$$|(T(\lambda)x, x)| \le \varepsilon \quad \Rightarrow \quad (T'(\lambda)x, x) \le -\delta.$$

Theorem (continued).
Let T be differentiable and assume in addition that the Virozub–Matsaev (VM) condition is satisfied:

$\exists\, \varepsilon, \delta > 0$ so that for all $x \in \mathcal{D}$, $\|x\| = 1$, and $\lambda \in (\alpha, \beta)$,

$$|(T(\lambda)x, x)| \leq \varepsilon \quad \Rightarrow \quad (T'(\lambda)x, x) \leq -\delta.$$

If $\sigma_{\text{ess}}(T) \neq \emptyset$ and T has N eigenvalues below $\sigma_{\text{ess}}(T)$, then

$$\inf_{\substack{L \subset \mathcal{D} \\ \dim L = n+\kappa}} \; \sup_{\substack{x \in L \\ x \neq 0}} \; p(x) = \min \sigma_{\text{ess}}(T), \qquad n > N.$$

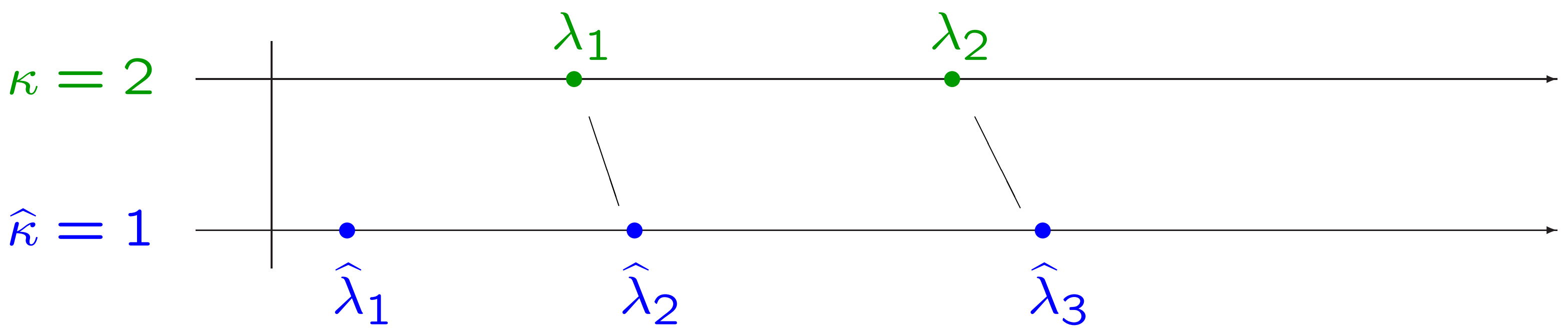
Comparison of two operator functions

Let T, $\widehat{T}$ be two operator functions as before such that $\mathcal{D} \supset \widehat{\mathcal{D}}$ and

$$p(x) \leq \widehat{p}(x), \qquad x \in \widehat{\mathcal{D}}.$$

Then

$$\kappa \geq \widehat{\kappa} \quad \text{and} \quad \lambda_{n-\kappa} \leq \widehat{\lambda}_{n-\widehat{\kappa}}.$$



Klein–Gordon equation

$$\left(-\left(-i\hbar\frac{\partial}{\partial t} - eq\right)^2 + c^2\left(-i\hbar\nabla - \frac{e}{c}\vec{A}\right)^2 + m^2c^4\right)U(x,t) = 0$$

This equation describes particles with spin 0, mass m and charge e.

q . . . electrostatic potential

$\vec{A}$ . . . electromagnetic potential

c . . . speed of light

$\vec{A}$ bounded on $\mathbb{R}^3$, $\quad |q(x)| \leq \dfrac{C}{\|x\|}$, C small enough.

Klein–Gordon equation

$$\left(-\left(-i\hbar\frac{\partial}{\partial t}-eq\right)^2+c^2\left(-i\hbar\nabla-\frac{e}{c}\vec{A}\right)^2+m^2c^4\right)U(x,t)=0$$

This equation describes particles with spin 0, mass m and charge e.

q . . . electrostatic potential

$\vec{A}$ . . . electromagnetic potential

c . . . speed of light

$\vec{A}$ bounded on $\mathbb{R}^3$, $\quad |q(x)| \le \frac{C}{\|x\|}$, C small enough.

$U(x,t)=e^{i\lambda t/\hbar}u(x)$, $\quad A_0=c^2(-i\hbar\nabla-\frac{e}{c}\vec{A})^2$, $\quad \gamma=mc^2$, $\quad V=eq$

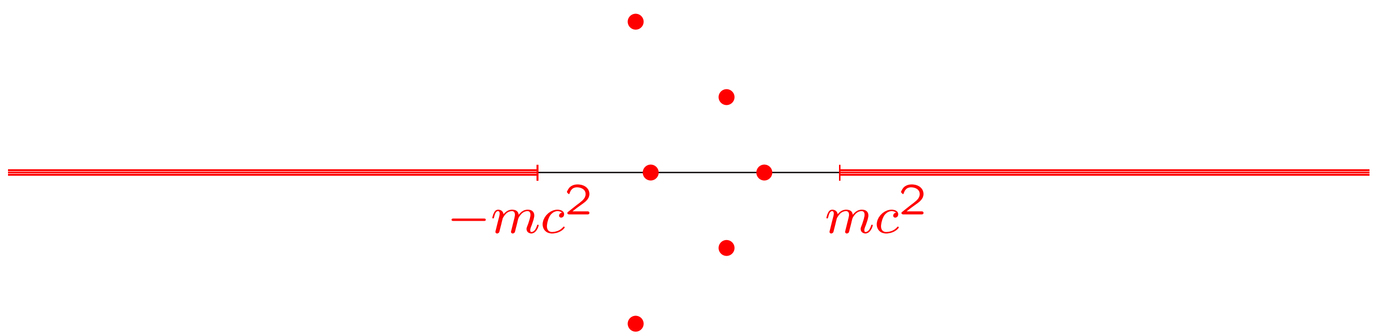
$$\rightsquigarrow \underbrace{\left(-(\lambda-V)^2+A_0+\gamma^2\right)}_{T(\lambda)}u=0$$

T operator function in $L^2(\mathbb{R}^3)$.

Spectrum of T

$$\sigma_{\mathsf{ess}}(T) = (-\infty, -mc^2] \cup [mc^2, \infty).$$

The non-real spectrum consists only of finitely many complex conjugate pairs of eigenvalues.



Generalised Rayleigh functional

For $u \in H^1(\mathbb{R}^3)$, $u \not\equiv 0$, let $p_\pm(u)$ be the zeros of $\lambda \mapsto (T(\lambda)u, u)$ if these are real;

otherwise, set $p_+(u) = -\infty$, $p_-(u) = +\infty$.

$$p_\pm(u) = (Vu, u) \pm \sqrt{(Vu, u)^2 - \|Vu\|^2 + \|(A_0 + \gamma^2)^{1/2} u\|^2}$$

$$\text{if } \|u\| = 1 \quad \text{and} \quad p_\pm(Cu) = p_\pm(u),\ C \neq 0$$

Theorem. [M.L., Tretter 2006]
Set $\nu := \sup\{p_-(u) : x \in H^1(\mathbb{R}^3),\ p_-(u) \neq +\infty\}$
and assume that $\nu < mc^2$.

Let $\alpha \in (\nu, \lambda_{\mathrm{e}})$ and κ be the number of neg. eigenvalues of $T(\alpha)$.

Then $\sigma(T) \cap (\alpha, \lambda_{\mathrm{e}})$ consists of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and

$$\lambda_n = \min_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n+\kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u).$$

Theorem. [M.L., Tretter 2006]
Set $\nu := \sup\{p_-(u) : x \in H^1(\mathbb{R}^3),\, p_-(u) \neq +\infty\}$
and assume that $\nu < mc^2$.

Let $\alpha \in (\nu, \lambda_{\mathrm{e}})$ and κ be the number of neg. eigenvalues of $T(\alpha)$.

Then $\sigma(T) \cap (\alpha, \lambda_{\mathrm{e}})$ consists of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and

$$\lambda_n = \min_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n+\kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u).$$

If there are infinitely many eigenvalues, then $\lim_{n\to\infty} \lambda_n = mc^2$.
If there are $N < \infty$ eigenvalues, then

$$\inf_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n+\kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u) = mc^2 \quad \text{for } n > N$$

Theorem. [M.L., Tretter 2006]
Set $\nu := \sup\{p_-(u) : x \in H^1(\mathbb{R}^3),\, p_-(u) \neq +\infty\}$
and assume that $\nu < mc^2$.

Let $\alpha \in (\nu, \lambda_e)$ and κ be the number of neg. eigenvalues of $T(\alpha)$.

Then $\sigma(T) \cap (\alpha, \lambda_e)$ consists of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and

$$\lambda_n = \min_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n+\kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u).$$

If there are infinitely many eigenvalues, then $\lim_{n\to\infty} \lambda_n = mc^2$.
If there are $N < \infty$ eigenvalues, then

$$\inf_{\substack{L \subset H^1(\mathbb{R}^3) \\ \dim L = n+\kappa}} \max_{\substack{u \in L \\ u \neq 0}} p_+(u) = mc^2 \quad \text{for } n > N$$

pairs of non-real eigenvalues $\leq \kappa$

In a similar way one can characterise eigenvalues $< \alpha$ with p_-.

Comparison with eigenvalues of a Schrödinger operator

Consider the corresponding Schrödinger operator

$$H = -\frac{1}{2m}\Big(-i\,\hbar\,\nabla - \frac{e}{c}\vec{A}\Big)^2 + V$$

and its negative eigenvalues $\mu_1 \leq \mu_2 \leq \cdots$. Then

$$\lambda^+_{n-\kappa_+} - mc^2 \leq \mu_n.$$

If $\vec{A} = 0$, then

$$\lim_{c\to\infty} (\lambda^+_{n-\kappa_+} - mc^2) = \mu_n.$$

Triple variational principle

Let T be defined on $[\alpha, \beta)$ with $\mathcal{D} = \text{dom}(T(\lambda))$ and $\alpha \in \rho(T)$. Set

$$\mathbf{M}_\alpha^+ := \Big\{\mathcal{M} \subset \mathcal{D} \colon \mathcal{M} \text{ is a maximal } (T(\alpha)\cdot,\cdot)\text{-non-negative subspace of } \mathcal{D}\Big\}.$$

$\mathcal{M}$ is $(T(\alpha)\cdot,\cdot)$-non-negative if $(T(\alpha)x, x) \geq 0$ for all $x \in \mathcal{M}$.

Triple variational principle

Let T be defined on $[\alpha, \beta)$ with $\mathcal{D} = \mathrm{dom}(T(\lambda))$ and $\alpha \in \rho(T)$. Set

$$\mathbf{M}_\alpha^+ := \Big\{\mathcal{M} \subset \mathcal{D}\colon \mathcal{M} \text{ is a maximal } (T(\alpha)\,\cdot,\cdot)\text{-non-negative subspace of } \mathcal{D}\Big\}.$$

$\mathcal{M}$ is $(T(\alpha)\,\cdot,\cdot)$-non-negative if $(T(\alpha)x, x) \geq 0$ for all $x \in \mathcal{M}$.

Theorem. [M.L., Strauss; cf. also Eschwé, H. Langer 2002]
Assume that the functions $(T(\cdot)x, x)$ are continuous and decreasing at value 0 for all $x \in \mathcal{D}$, $x \neq 0$.
Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be eigenvalues of T in $(\alpha, \min \sigma_{\mathrm{ess}}(T))$. Then

$$\lambda_n \geq \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \ \sup_{\substack{L \subset \mathcal{M} \\ \dim L = n-1}} \ \inf_{\substack{x \perp L \\ x \in \mathcal{M},\, x \neq 0}} p(x).$$

Theorem (continued).
Let T be differentiable and assume in addition that the Virozub–Matsaev (VM) condition is satisfied. Then

$$\min \sigma_{\mathrm{ess}}(T) \geq \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \sup_{\substack{L \subset \mathcal{M} \\ \dim L = n-1}} \inf_{\substack{x \perp L \\ x \in \mathcal{M},\, x \neq 0}} p(x), \qquad n = 1, 2, \ldots .$$

In particular,

$$\min \sigma(T) \geq \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \inf_{x \in \mathcal{M},\, x \neq 0} p(x).$$

Theorem (continued).
Let T be differentiable and assume in addition that the Virozub–Matsaev (VM) condition is satisfied. Then

$$\min \sigma_{\mathrm{ess}}(T) \geq \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \sup_{\substack{L \subset \mathcal{M} \\ \dim L = n-1}} \inf_{\substack{x \perp L \\ x \in \mathcal{M},\, x \neq 0}} p(x), \qquad n = 1, 2, \ldots.$$

In particular,

$$\min \sigma(T) \geq \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \inf_{x \in \mathcal{M},\, x \neq 0} p(x).$$

Corollary. Let $\mu \in (\alpha, \beta)$ and assume that (VM) is satisfied. If there exists a maximal $(T(\alpha)\cdot,\cdot)$-non-negative subspace $\mathcal{M}$ such that

$$(T(\mu)x, x) \geq 0, \qquad \text{for all } x \in \mathcal{M},$$

then $[\alpha, \mu) \subset \rho(T)$.

Theorem. [M.L., Strauss]
If T is an analytic operator function whose values are bounded operators and which satisfies the (VM) condition, then

$$\lambda_n = \sup_{\mathcal{M} \in \mathbf{M}_\alpha^+} \sup_{\substack{L \subset \mathcal{M} \\ \dim L = n-1}} \inf_{\substack{x \perp L \\ x \in \mathcal{D},\, x \neq 0}} p(x).$$

Idea of proof: choose spectral subspace for the operator function obtained from a linearisation of T in [H. Langer, Markus, Matsaev 2000, 2006].

Block operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

where A and D are self-adjoint operators.

Introduce the Schur complement

$$T(\lambda) = A - \lambda - B(D - \lambda)^{-1}B^*, \qquad \lambda \in \rho(D).$$

On $\rho(D)$ the spectra of $\mathcal{A}$ and T coincide, i.e.

$$\sigma(\mathcal{A}) \cap \rho(D) = \sigma(T).$$

Theorem. [M.L., Strauss]

Assume that D is bounded with $d_+ := \max \sigma(D)$,

A is bounded from below, $\mathrm{dom}(|A|^{1/2}) \subset \mathrm{dom}(B^*)$ and

$$\|B^* x\|^2 \leq a\|x\|^2 + b(Ax, x), \qquad x \in \mathrm{dom}(A).$$

for some $a, b \geq 0$.

Let $d_+ < \alpha < \beta$, $(\alpha, \beta) \subset \rho(A)$ and set

$$\hat{\alpha} := \frac{\alpha + d_+}{2} + \sqrt{\left(\frac{\alpha - d_+}{2}\right)^2 + a + b\alpha}$$

If $\hat{\alpha} < \beta$, then $(\hat{\alpha}, \beta) \subset \rho(\mathcal{A})$.

Moreover, equality in variational principle holds for eigenvalues in certain gaps of the essential spectrum.

Theorem. [M.L., Strauss]

$\mathcal{A}$ as above.

Let $d_+ < \alpha < \beta$, and assume that $(\alpha, \beta) \cap \sigma_{\mathrm{ess}}(\mathcal{A}) = \emptyset$. Set

$$\hat{\alpha} := \frac{\alpha + d_+}{2} + \sqrt{\left(\frac{\alpha - d_+}{2}\right)^2 + a + b\alpha}$$

If $\hat{\alpha} < \beta$, then $(\hat{\alpha}, \beta) \cap \sigma_{\mathrm{ess}}(\mathcal{A}) = \emptyset$.