
Solving Rational Eigenvalue Problems via Linearization

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MOP Meeting 2, University of Nottingham, March 29-30, 2010

1. Introduction: REPs in NEPs	2
2. Case I: Loaded Elastic String	8
3. Case II: Fluid-Solid Vibration	12
4. Case III: QEP	17
5. General cases, with applications	18

- SEP: $(A - \lambda I)x = 0$
- GEP: $(A - \lambda B)x = 0$
- QEP: $(\lambda^2 M + \lambda D + K)x = 0$
- PEP:

$$\left(\sum_{i=0}^d \lambda^i A_i \right) x = 0$$

- REP:

$$\left(A - \lambda B - \sum_{i=1}^k \frac{\lambda}{\lambda - \sigma_i} C_i \right) x = 0$$

- NEP:

$$T(\lambda)x = 0$$

QEP:

$$\left(\lambda^2 M + \lambda D + K \right) x = 0$$

Linearization: first canonical form

$$\left(\lambda \begin{bmatrix} M & \\ & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & \end{bmatrix} \right) \begin{bmatrix} \lambda x \\ x \end{bmatrix} = 0$$

- Many linearizations
- MATLAB function `polyeig` uses the 1st canonical form and then call `eig`, by Higham & Tisseur
- a very active research topic

Picard iteration: given λ_0 ,

$$(A - \lambda_{k+1}B - s(\lambda_k)E)x = 0, \quad k = 0, 1, 2, \dots$$

Newton method:

$$\begin{cases} T(\lambda_k)x_{k+1} + (\lambda_{k+1} - \lambda_k)T'(\lambda_k)x_k = 0 \\ c^T x_{k+1} - 1 = 0 \end{cases}$$

Successive linearization approximation method (SLAM):

$$\begin{cases} [T(\lambda_k) + (\lambda_{k+1} - \lambda_k)T'(\lambda_k)]x_{k+1} = 0 \\ c^T x_{k+1} - 1 = 0 \end{cases}$$

- Nonlinear inverse iteration
- Nonlinear Rayleigh Quotient iteration
- Nonlinear Arnoldi method
- Nonlinear Jacobi-Davidson algorithm
- Nonlinear conjugate gradient method, . . .

Problems:

- global convergence?
- convergent to a desired eigenvalue?
- all desired are computed? any lost?
- structure preservation and utilization?

$$R(\lambda)x = \left(P(\lambda) - \sum_{i=1}^k \frac{s_i(\lambda)}{q_i(\lambda)} E_i \right) x = 0$$

- Brute-force way: Multiply with $\prod_{i=1}^k q_i(\lambda)$ to get a PEP
 - very high degree if k is large, e.g. $k = 9$
 - all matrices mixed
 - structures?
- Looking REPs as NEPs
 - nonlinear methods
 - Convergence? Any lost?

Via linearization,

- equivalence: no eigenvalue lost
- easy computation: standard algorithms and software, e.g. ARPACK

New linearization vs brute-force way + PEP linearization

- smaller-size LEP
- structure preservation
 - ↪ properties preservation
 - ↪ better algorithms
 - ↪ faster convergence

REP from loaded elastic string:

$$\left(A - \lambda B + \frac{\lambda}{\lambda - 1} e_n e_n^T \right) x = 0,$$

- A, B : SPD, tridiagonal
- e_n : the last column of identity

S. I. Solov'ëv. Preconditioned iterative methods for a class of nonlinear eigenvalue problems. *Linear Algebra Appl.*, 415:210-229, 2006

T. Betcke and N. J. Higham and V. Mehrmann and C. Schroder and F. Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. <http://www.manchester.ac.uk/mims/eprints>, 2008.

REP:

$$\left(A - \lambda B + \frac{\lambda}{\lambda - 1} e_n e_n^T \right) x = 0,$$

Rewrite in proper form

$$(A + e_n e_n)x - \lambda Bx + \underline{e_n e_n^T x / (\lambda - 1)} = 0$$

New variable:

$$y = e_n^T x / (\lambda - 1) \quad \Leftrightarrow \quad e_n^T x + y - \lambda y = 0$$

Linearization of REP:

$$(\mathcal{A} - \lambda \mathcal{B})z = \left(\begin{bmatrix} A + e_n e_n^T & e_n \\ e_n^T & 1 \end{bmatrix} - \lambda \begin{bmatrix} B & \\ & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

- Structure preservation: \mathcal{A} and \mathcal{B} are SPD and tridiagonal

10 computed smallest eigenvalues by MATLAB function eig for $n = 100$:

i	$\hat{\lambda}_i$	residual
1	0.457318488953671	$5.58e - 013$
2	4.48217654587198	$5.96e - 013$
3	24.2235731125539	$6.69e - 013$
4	63.7238211419405	$9.40e - 013$
5	123.031221067605	$8.63e - 013$
6	202.200899143561	$9.56e - 013$
7	301.310162794155	$1.09e - 012$
8	420.456563106511	$1.01e - 012$
9	559.757586307048	$7.12e - 013$
10	719.350660116386	$9.15e - 013$

Full precision!

 CPU time

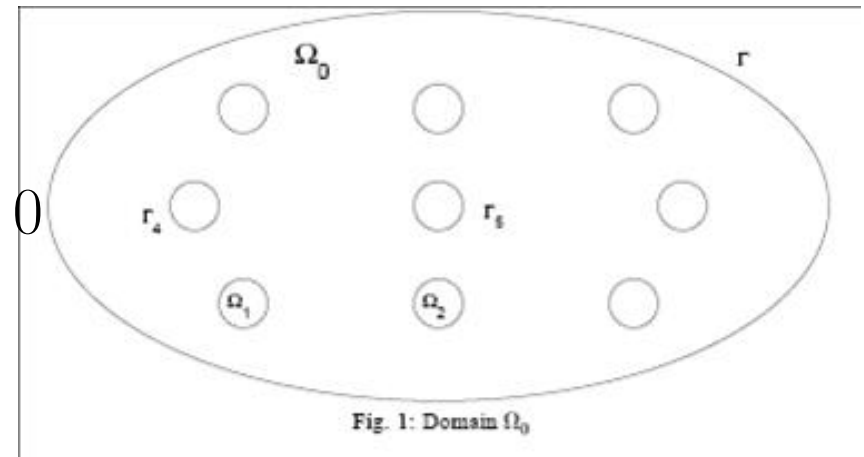
	solver	$n = 200$	$n = 400$	$n = 600$	$n = 800$
$A - \lambda B$	eig	0.0156	0.1406	1.0625	3.5469
QEP	polyeig	0.7500	6.8594	24.5313	84.9063
$A - \lambda B$	eig	0.0156	0.1406	1.0469	3.5313

with new linearization, solving this REP is as easy as solving an LEP !

Rational Eigenvalue Problem

$$\left(A - \lambda B - \sum_{i=1}^k \frac{\lambda}{\sigma_i - \lambda} C_i C_i^T \right) x = 0$$

- A, B : SPD; C_i : low rank; $\sigma_i > 0$



An example

- number of rational terms $k = 9$; poles $\sigma_i = i, i = 1 : 9$;
- C_i : $n \times 2$
- size $n = 36,046$;
- interested eigenvalues: all ones in $(\sigma_1, \sigma_2) = (1, 2)$.

H. Voss. A rational spectral problem in fluid-solid vibration. *E. Trans. Numer. Anal.* vol 16, pp 94-106, 2003.

REP:

$$\left(A - \lambda B + \sum_{i=1}^k \frac{\lambda}{\lambda - \sigma_i} C_i C_i^T \right) x = 0,$$

Rewrite in proper form:

$$\left(A + \sum_{i=1}^k C_i C_i^T - \lambda B + \sum_{i=1}^k \frac{\sigma_i}{\lambda - \sigma_i} C_i C_i^T \right) x = 0,$$

New variables:

$$y_i = \sigma_i C_i^T x / (\lambda - \sigma_i) \quad \Leftrightarrow \quad C_i^T x + y_i - \lambda y_i / \sigma_i = 0$$

Linearization of REP:

$$(\mathcal{A} - \lambda \mathcal{B})z = \left(\begin{bmatrix} \tilde{A} & C_1 & \cdots & C_k \\ C_1 & I_2 & & \\ \vdots & & \cdots & \\ C_k & & & I_2 \end{bmatrix} - \lambda \begin{bmatrix} B & & & \\ & I_2/\sigma_1 & & \\ & & \cdots & \\ & & & I_2/\sigma_k \end{bmatrix} \right) \begin{bmatrix} x \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = 0.$$

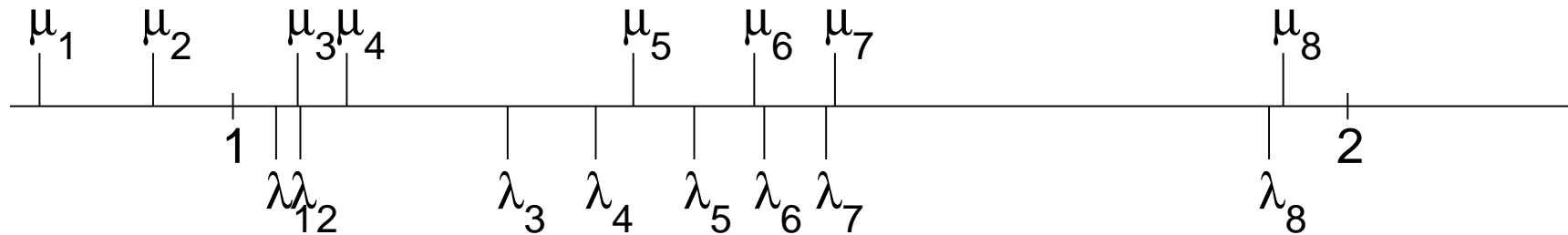
After new linearization

- Size after new linearization $36,046 + 2 \times 9 = 36064$;
 - Structure preserved: \mathcal{A} is symmetric, \mathcal{B} is positive definite
- ? \mathcal{A} may be not sparse, however, its is a sparse matrix with low rank updating!

CPU time

	$A - \lambda B$	$A - \lambda B$
ldlsparse	0.64	0.64
preprocessing	0.49	0
eigs	7.14	6.66
Total	8.27	7.30

With new linearization, solving this REP is as easy as solving an LEP!



- $\lambda_1, \dots, \lambda_8$: Linearized REP $A - \lambda B$;
- μ_1, \dots, μ_8 : $A - \lambda B$: linear part without rational terms.

QEP:

$$(\lambda^2 M + \lambda D + K)x = 0$$

Let $K = LR$: LU, QR, Cholesky, or Rank-revealing Decomp, \dots

$$\lambda Mx + Dx + \underline{L(Rx)/\lambda} = 0$$

New variable $y = Rx/\lambda$. “Linearization” of QEP:

$$\left(\lambda \begin{bmatrix} M & \\ & -I \end{bmatrix} + \begin{bmatrix} D & L \\ R & \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

- 1st canonical form: $L = K, R = I$;
- 2nd canonical form: $L = I, R = K$;
- If L, R are of low ranks, the size is smaller than $2n$. More importantly, those explicit zero eigenvalues are kicked away!

REP:

$$R(\lambda)x = \left(P(\lambda) - \sum_{i=1}^k \frac{s_i(\lambda)}{q_i(\lambda)} E_i \right) x = 0.$$

- more complex linearization, involving minimal realization of rational functions, etc
- more applications
- more details
- **Key:** E_i is of low rank!

Yangfeng Su and Zhaojun Bai. Solving Rational Eigenvalue Problems via Linearization.

<http://www.cs.ucdavis.edu/research/tech-reports/2008/CSE-2008-13.pdf> Submitted to SIMAX

Thank you for your attention!