An operator pencil in coastal oceanography: observations, numerics, proof, asymptotics.

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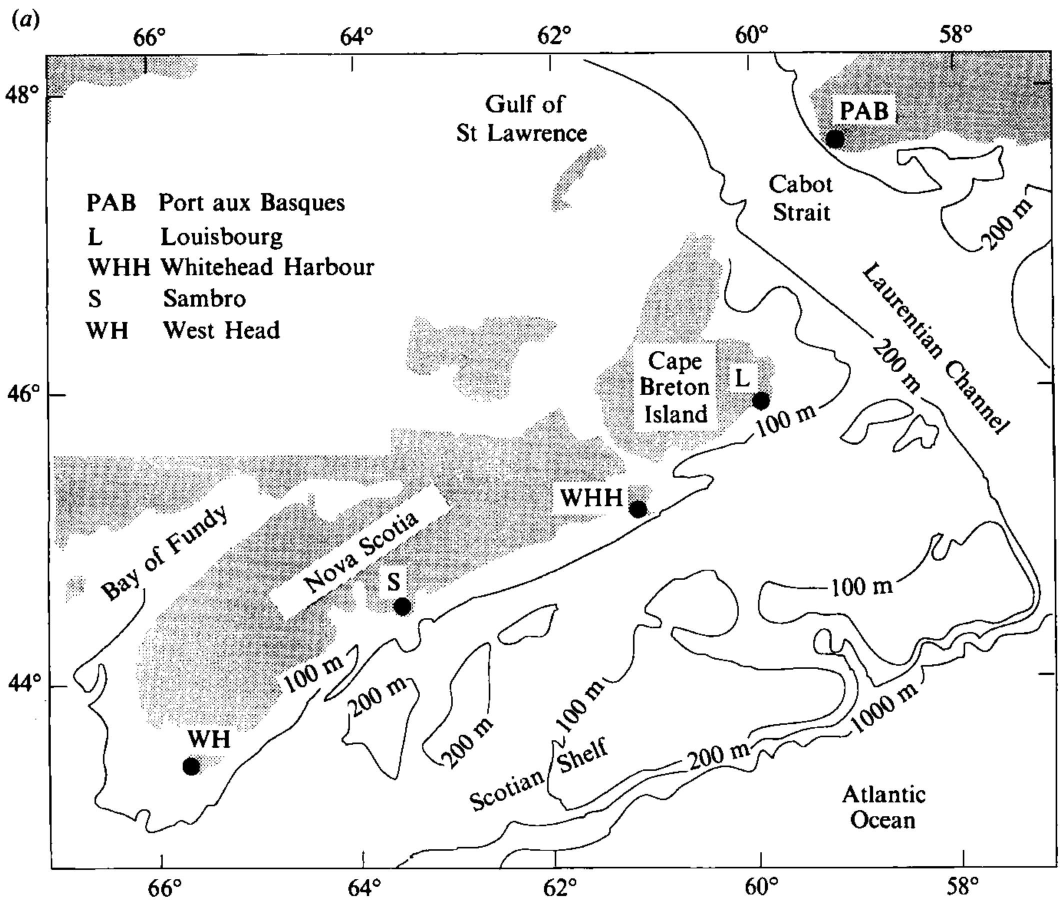
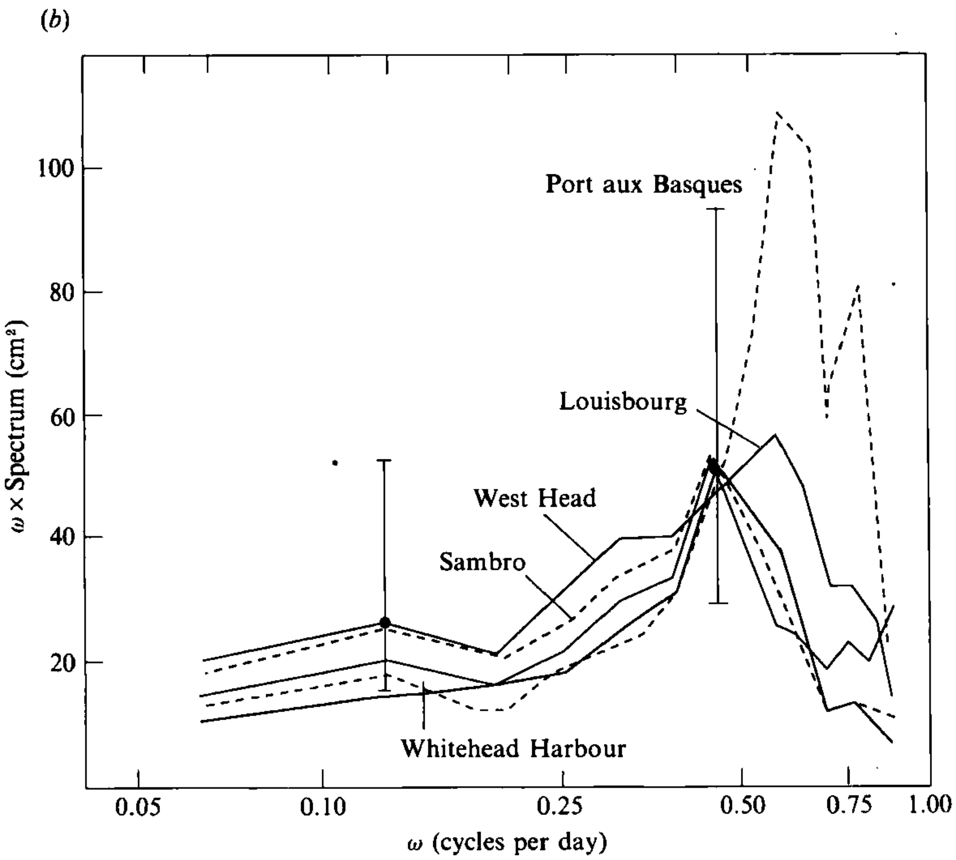
Scotian Shelf



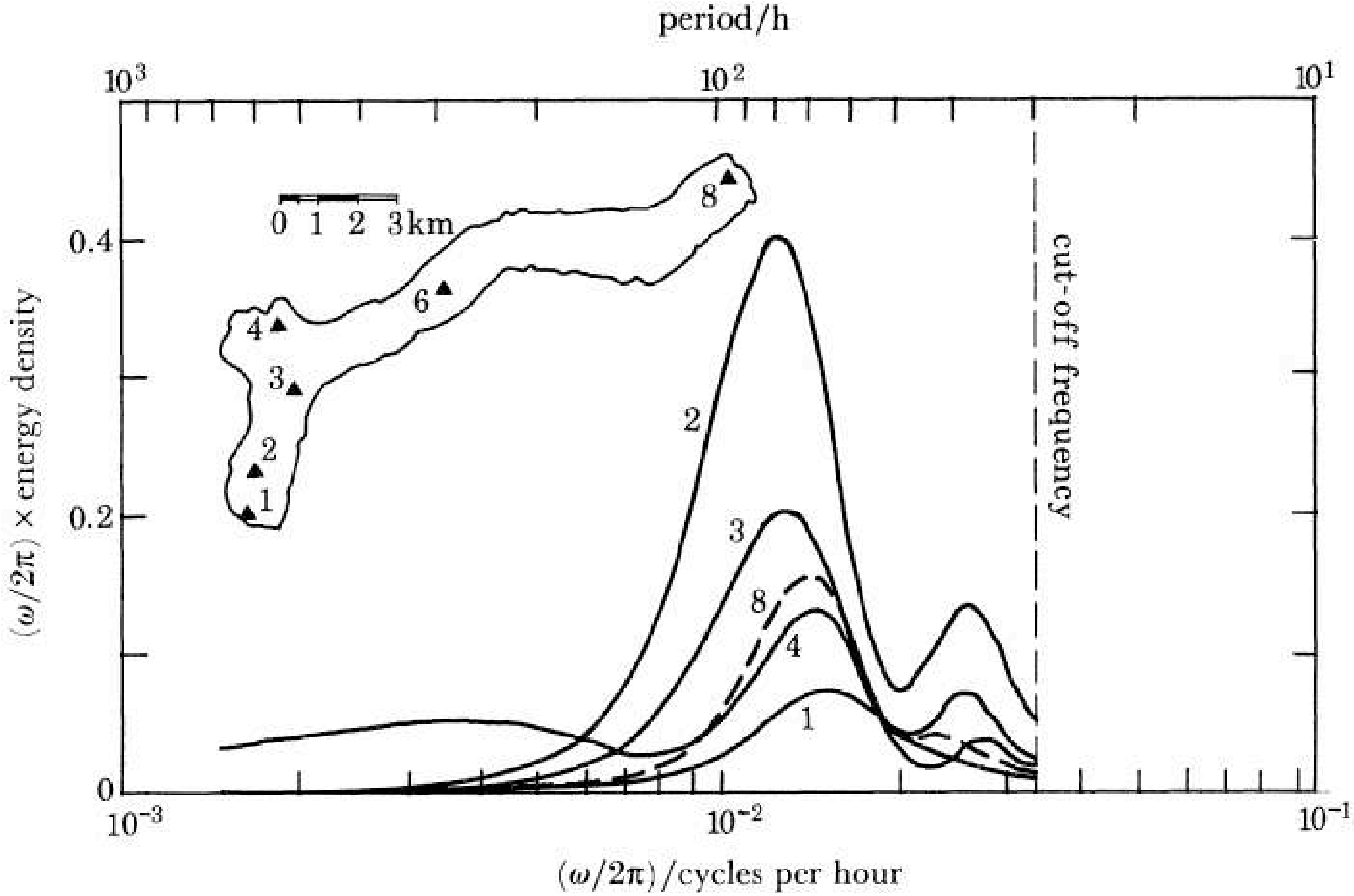
Tide-gauge stations from Schwing 1989

Scotian Shelf



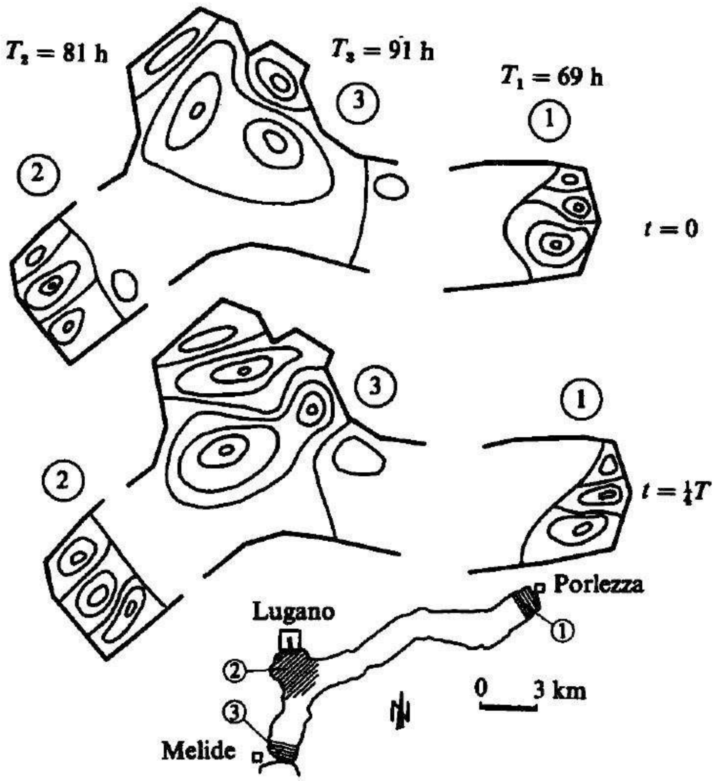
Autospectra of filtered time series from Schwing 1989. Strongest response at Port aux Basques in Cabot Strait and at Louisbourg on shelf near mouth of Laurentian Channel has frequency above that of strongest propagating wave at Whitehead Harbour, Sambro and West Head away from channel mouth.

Lake of Lugano



Lake of Lugano spectra from Mysak, Salvadé, Hutter and Scheiwiller, 1987

Lake of Lugano



Trösch 1984 : finite element *barotropic* model obtained Bay modes.

Governing equation

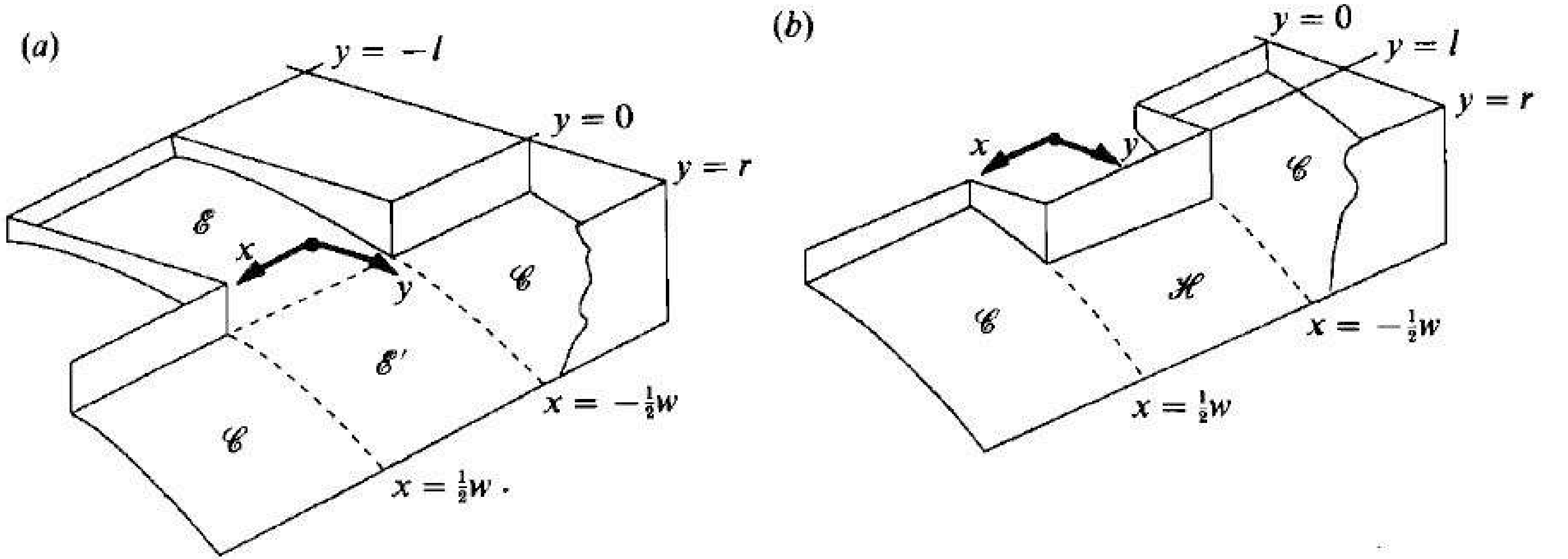
- The governing equation is taken to be the constant f barotropic shelf wave equation:

$$\nabla \cdot (H^{-1} \nabla \Psi_t) + f(\nabla \Psi \times \nabla H^{-1}) \cdot \hat{z} = 0.$$

- Here $H(x, y)$ is the (given) local undisturbed fluid depth, Ψ is the volume flux streamfunction and the velocity is $\boldsymbol{u} = H^{-1} \hat{\boldsymbol{z}} \times \nabla \Psi$, $\hat{\boldsymbol{z}}$ a unit vertical vector.
- Writing $\Psi(x, y, t) = \Re\{\exp(-\mathrm{i}\omega f t)\psi(x, y)\}$ gives the pencil

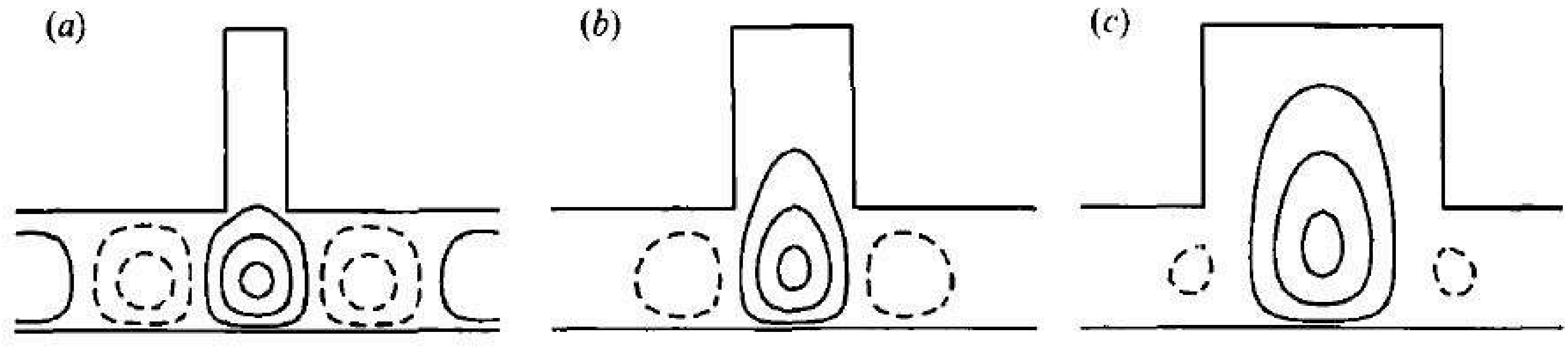
$$\nabla \cdot (H^{-1} \nabla \psi) + \mathrm{i}\omega^{-1}(\nabla \psi \times \nabla H^{-1}) \cdot \hat{z} = 0.$$

An estuary/headland model



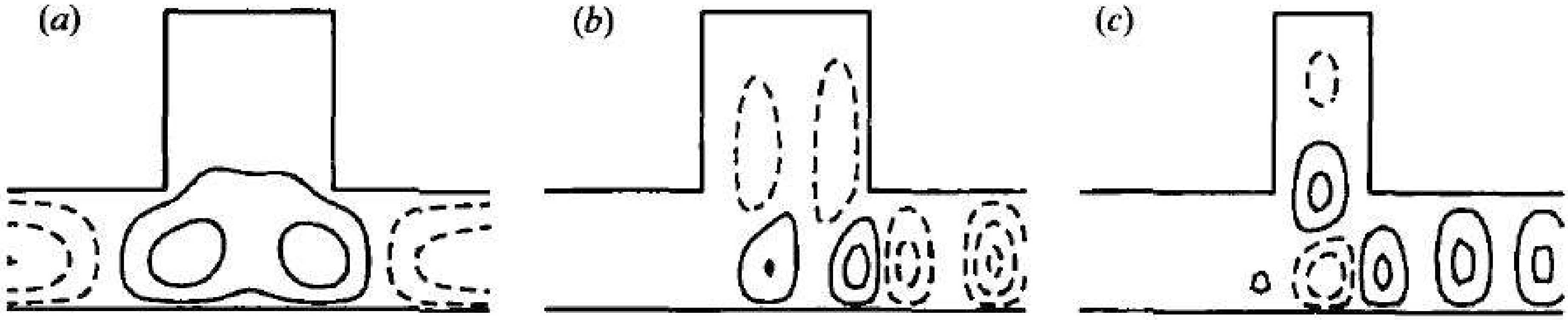
(Stocker & J. 1991)

Estuary-trapped modes



- The estuary is wider.
- Modes propagate there at higher frequencies than on the shelf.
- These form trapped modes.

Perfect transmission and reflection



Conformal invariance

- The governing equation

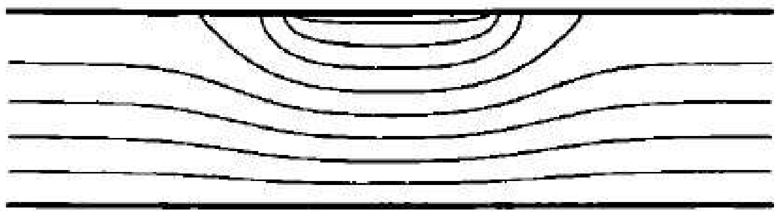
$$\nabla \cdot (H^{-1}\nabla\psi_t) + f(\nabla\psi \times \nabla H^{-1}) \cdot \hat{z} = 0.$$

  is invariant under conformal mappings.

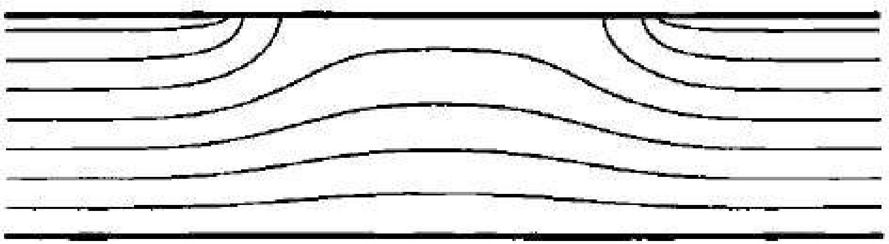
- Can form the basis for constructing solutions

- But means that any shelf that can be conformally mapped to a straight shelf cannot support trapped modes.

Conformally-equivalent channels

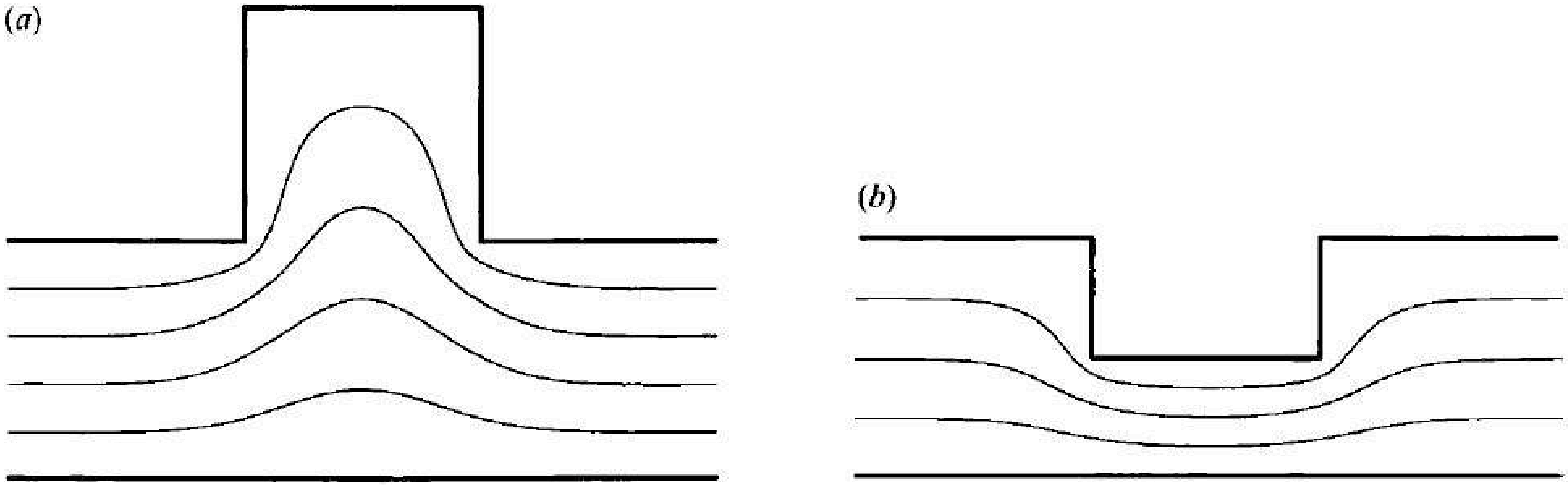


Bay equivalent
–trapping–

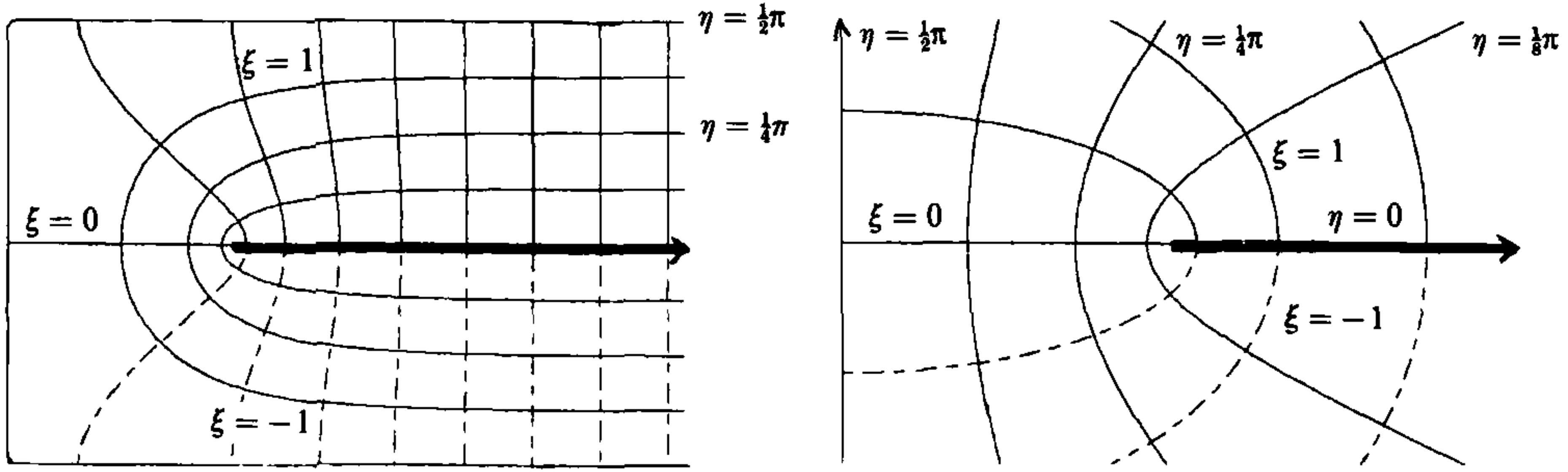


Headland equivalent
–no trapping–

All frequency transmission



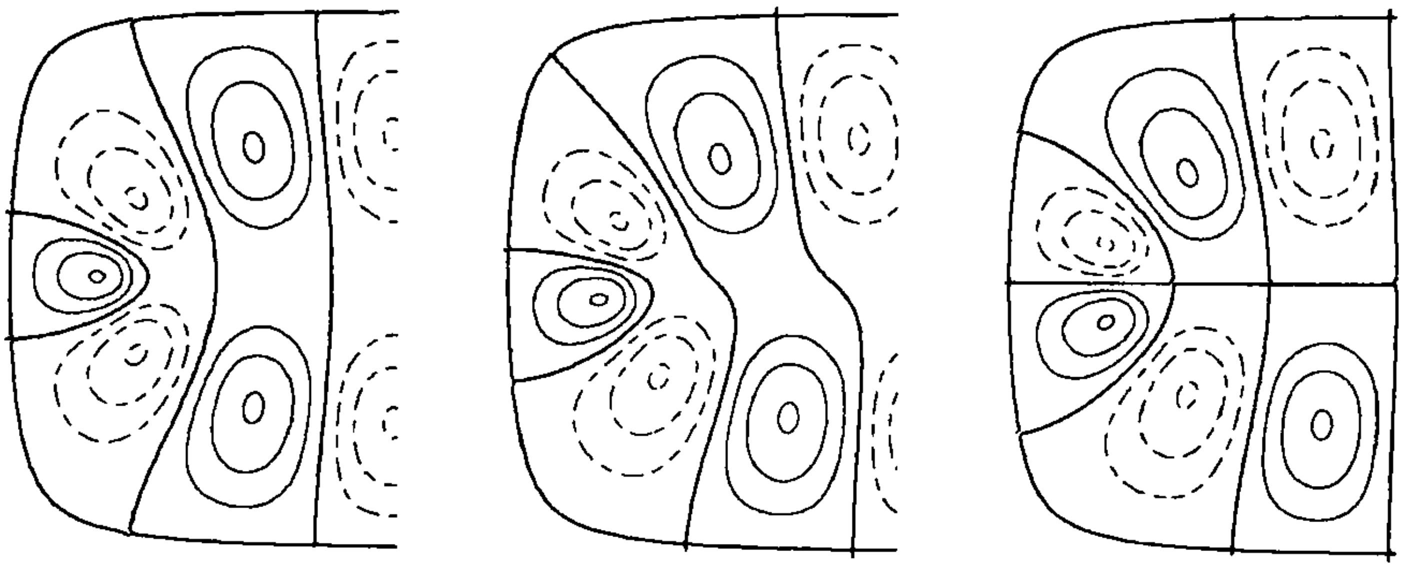
A semi-infinite channel



The equivalent co-ordinate systems. The depth is a function of η alone.

Cut must be treated with care: solutions in Mysak *et al.* incorrect.

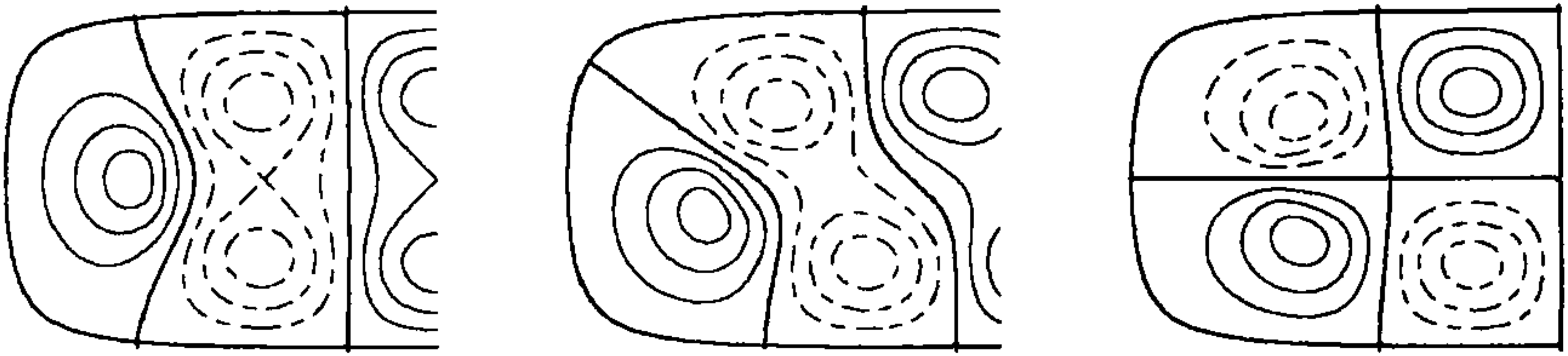
A semi-infinite channel



Tightly bunched end contours

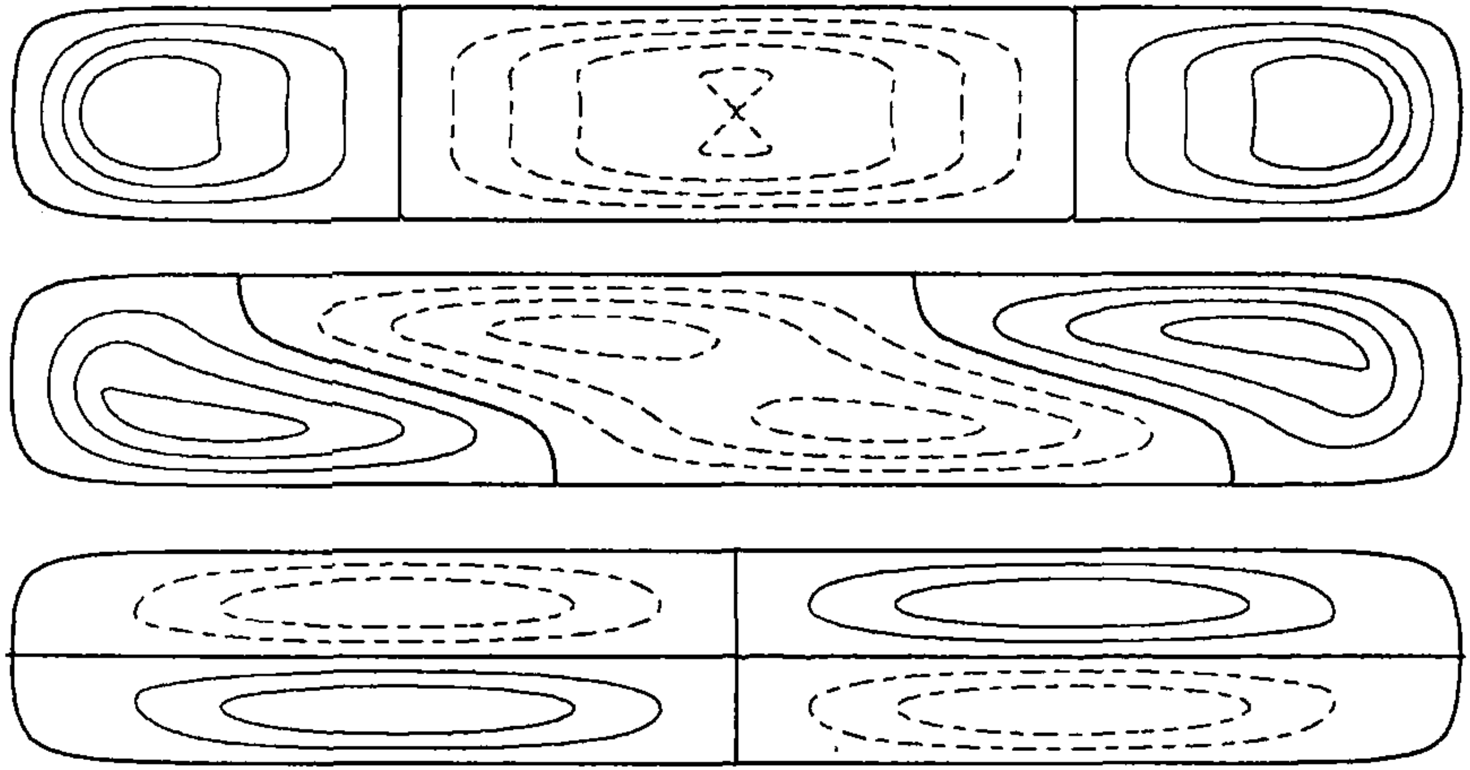
(J. 1987)

A semi-infinite channel

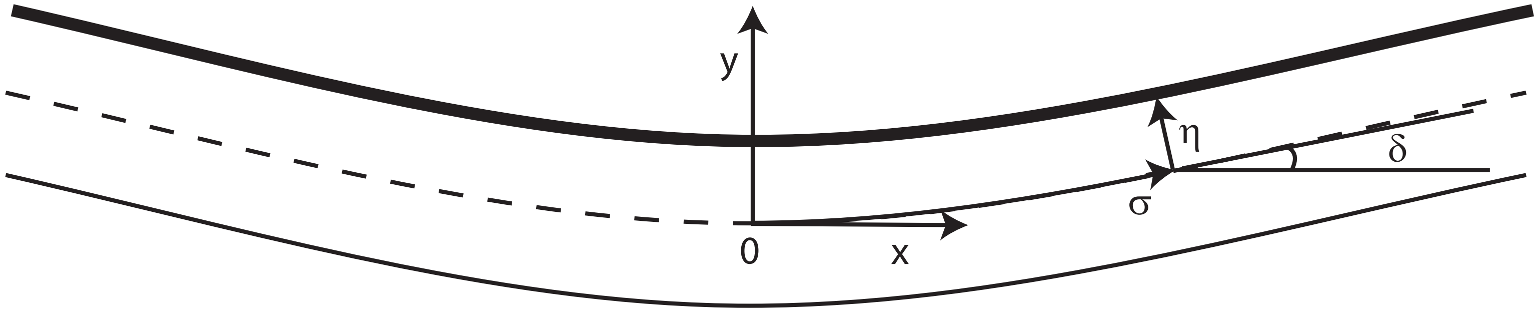


Smoothly turning channel end

Whole lake modes



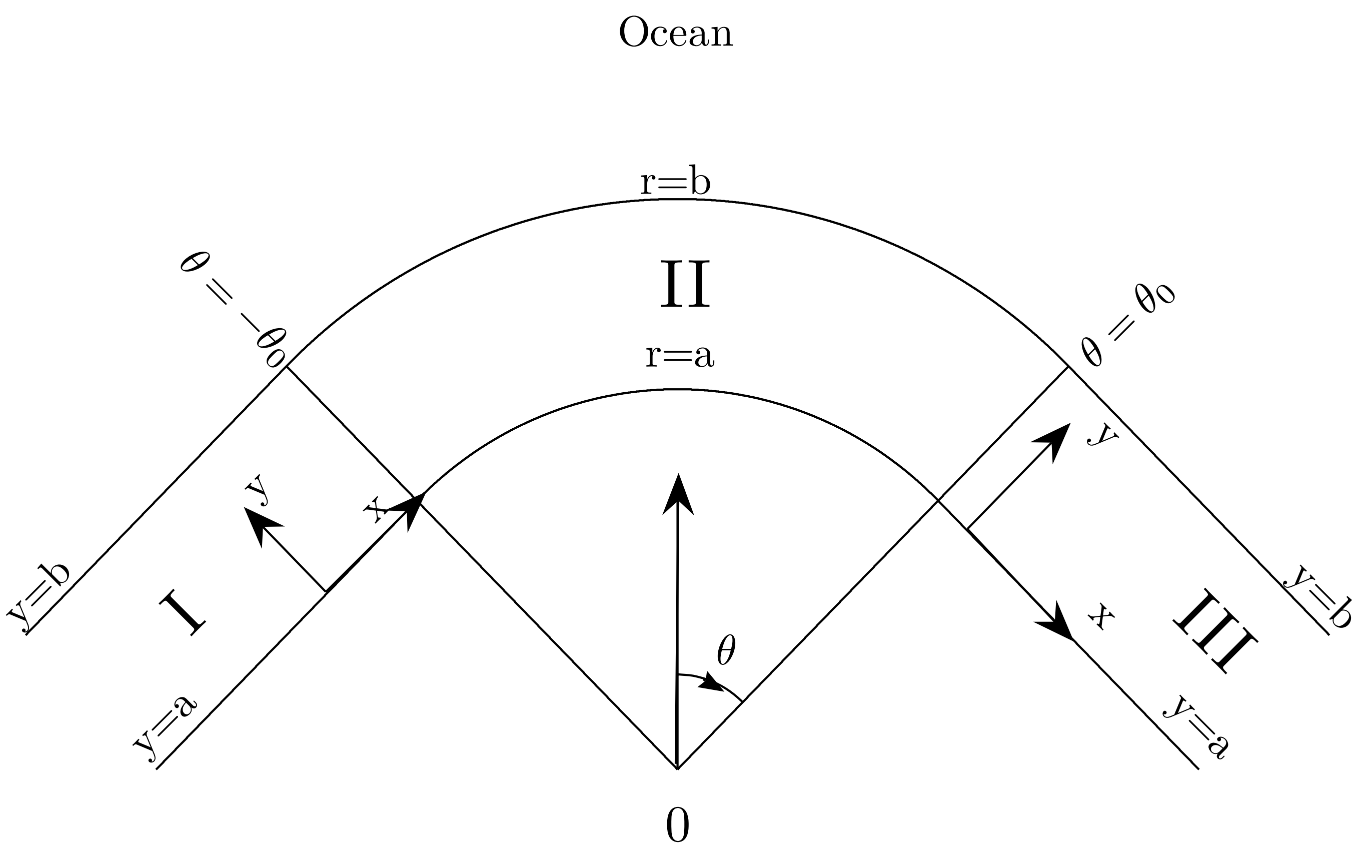
Coastal curvature



(J., Levitin and Parnovski 2006)

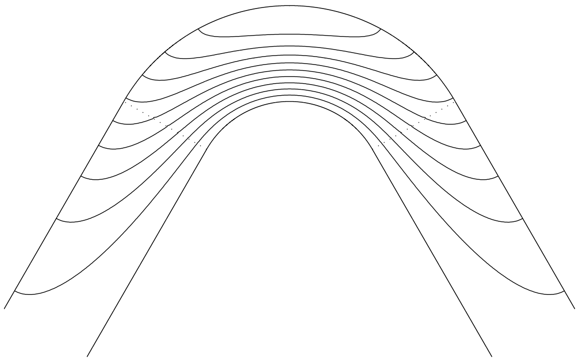
- *If one takes a rectilinear smooth shelf supporting only propagating modes and bends it slightly to form a cape then it supports a trapped mode*
- Not so with bay
- Does not contradict conformal mapping result

Coastal curvature – mode matching

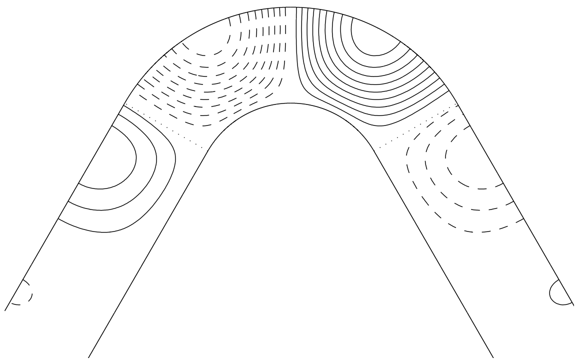


(Kaoullas & J. 2011)

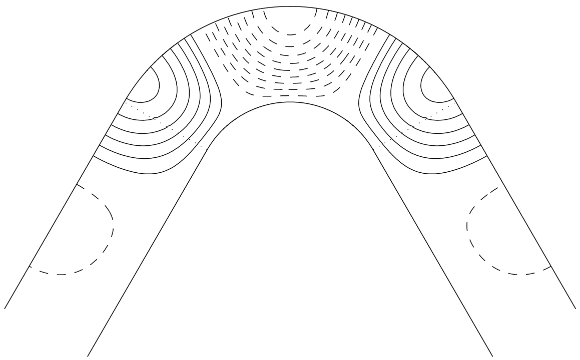
Coastal curvature – lowest mode



$|\psi|$



$\Re\{\psi\}$



$\Im\{\psi\}$

Coastal curvature – weak curvature

Postnova & Craster, 2008

- Take $H(\eta) = \exp[-2b(1+\eta)]$ $(b > 0)$ (Buchwald & Adams, 1968)
- Write $\xi = \epsilon\sigma$ $(0 < \epsilon \ll 1)$, slowly-varying
- Write $\Psi(x, y, t) = \Re\{\exp[-\mathrm{i}(\omega f t + b\sigma/\omega)]\Phi(\xi, \eta)\}$, removing fast phase so Φ is slowly-varying envelope.
- Write $p = (1 - \epsilon\delta_\xi \eta)^{-1}$, curvature function

Coastal curvature – weak curvature equation

Then

$$\epsilon^2 p^2 \Phi_{\xi\xi} + \Phi_{\eta\eta} + [\epsilon^3 p^3 \eta \delta_{\xi\xi} - \mathrm{i}\epsilon^2 (2b/\omega) p^2 \eta \delta_\xi] \Phi_\xi + (2b - \epsilon p \delta_\xi) \Phi_\eta + \\ [(b/\omega)^2 (2p - p^2) - \mathrm{i}\epsilon^2 (b/\omega) p^3 \eta \delta_{\xi\xi}] \Phi = 0,$$

with boundary conditions

$$\Phi = 0, \qquad (\eta = 1), \qquad\qquad \Phi_\eta = 0, \qquad (\eta = -1).$$

Coastal curvature – weak curvature expansion

PC continue by introducing the regular expansion

$$\begin{aligned}\Phi(\xi,\eta) &= \Phi^{(0)}(\xi,\eta)+\epsilon\Phi^{(1)}(\xi,\eta)+\epsilon^2\Phi^{(2)}(\xi,\eta)+\ldots,\\ (b/\omega)^2 &= \mu_0+\epsilon\mu_1+\epsilon^2\mu_2+\epsilon^3\mu_3+\epsilon^4\mu_4+\ldots.\end{aligned}$$

The leading order system is then

$$\mathcal{L}\Phi^{(0)}=0, \qquad \text{where} \qquad \mathcal{L}\Phi=\Phi_{\eta\eta}+2b\Phi_\eta+\mu_0\Phi,$$

with boundary conditions $\Phi^{(0)}(\xi,1)=\Phi_\eta^{(0)}(\xi,-1)=0$, giving

$$\Phi^{(0)}=f^{(0)}(\xi)\mathrm{e}^{-b\eta}\sin\gamma(\eta-1), \qquad \gamma=\sqrt{\mu_0-b^2},$$

with μ_0 determined from

$$\tan 2\gamma=-\gamma/b,$$

and $f^{(0)}(\xi)$ to be determined.

Coastal curvature – weak curvature expansion

The first order system is

$$\mathcal{L}\Phi^{(1)} = \delta_\xi \Phi_\eta^{(0)} - \mu_1 \Phi^{(0)}, \tag{1}$$

with $\Phi^{(1)}(\xi, 1) = \Phi_\eta^{(1)}(\xi, -1) = 0$.

- Solvability: Right side orthogonal to solutions to the adjoint of the homogeneous problem.
- Operator $\mathcal{L}$ is not self-adjoint.
- Slip in PC – orthogonal to solution of original homogeneous problem

Equivalent and simpler here to write (1) in self-adjoint form

$$\widehat{\mathcal{L}}\Phi^{(1)} = \mathrm{e}^{2b\eta}(\delta_\xi \Phi_\eta^{(0)} - \mu_1 \Phi^{(0)}), \qquad \text{where} \qquad \widehat{\mathcal{L}}\Phi = (\mathrm{e}^{2b\eta}\Phi_\eta)_\eta + \mu_0 \mathrm{e}^{2b\eta}\Phi,$$

with boundary conditions unaltered.

Coastal curvature – weak curvature expansion

Take $\Phi^{(0)}$ as solution of the homogeneous adjoint. Then

$$\int_{-1}^{1} \mathrm{e}^{2b\eta}\Phi^{(0)}(\delta_\xi \Phi_\eta^{(0)} - \mu_1 \Phi^{(0)})\,\mathrm{d}\eta = 0, \qquad \text{i.e.} \qquad \mu_1 + \beta\delta_\xi = 0, \quad (2)$$

where

$$\beta = b + \gamma(1 - \cos 4\gamma)/(4\gamma - \sin 4\gamma),$$

is a positive number.

As in PC: δ_ξ varies with ξ, no choice of the number μ_1 satisfies (2) for all ξ.

Hence rescale (weaken) curvature: $\delta_\xi = \epsilon\hat{\delta}_\xi$. (2) is simply $\mu_1 = 0$.

Coastal curvature – weak curvature expansion

Second order, in self-adjoint form

$$\widehat{\mathcal{L}}\Phi^{(2)} = \mathrm{e}^{2b\eta}(-\Phi^{(0)}_{\xi\xi} + \hat{\delta}_\xi \Phi^{(0)}_\eta - \mu_2 \Phi^{(0)}),$$

with $\Phi^{(2)}(\xi, 1) = \Phi^{(2)}_\eta(\xi, -1) = 0$.
Choose $\Phi^{(0)}$ as adjoint solution.
Apply solvability condition to give

$$f^{(0)}_{\xi\xi} + (\mu_2 + \beta\hat{\delta}_\xi) f^{(0)} = 0.$$

– steady 1D Schrödinger equation (trapped mode = bound state)

Schrödinger (Landau & Lifshitz)

If $\hat{\delta}_\xi$ everywhere negative then no trapped mode.
Suppose $\hat{\delta}_\xi$ somewhere positive. The total change in direction of the shelf is $\epsilon\Delta$ where $\Delta = [\hat{\delta}]_{-\infty}^{\infty}$. Then for a shelf with a finite region of non-zero-curvature and sufficiently small but positive Δ, a trapped mode always exists:-
In the region of non-zero curvature μ_2 can be taken to be small compared to $\beta\hat{\delta}_\xi$ and $f^{(0)}$ taken to be a constant, which can be taken to be unity. Then, for $|\beta\Delta| \ll 1$,

$$f_{\xi\xi}^{(0)} = -\beta\hat{\delta}_\xi,$$

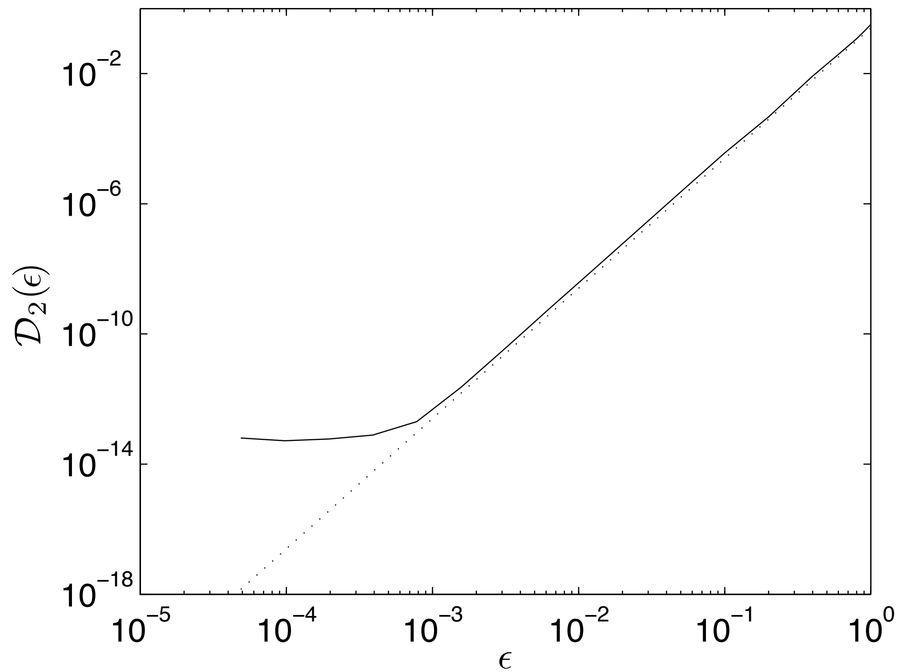
giving the change in $f_\xi^{(0)}$ across the region as

$$[f_\xi^{(0)}] = -\beta \int_{-\infty}^{\infty} \hat{\delta}_\xi \, \mathrm{d}\xi = -\beta\Delta.$$

At large distances $f^{(0)} = \exp\left[-\sqrt{-\mu_2}|\xi|\right]$ so $-2\sqrt{-\mu_2} = -\beta\Delta$.
Thus for trapped modes, in the limit $|\beta\Delta| \ll 1$, $\beta\Delta$ must be positive and μ_2 negative with

$$\mu_2 \to -\frac{1}{4}\beta^2\Delta^2, \text{ as } \beta\Delta \to 0^+.$$

Sech-squared curvature



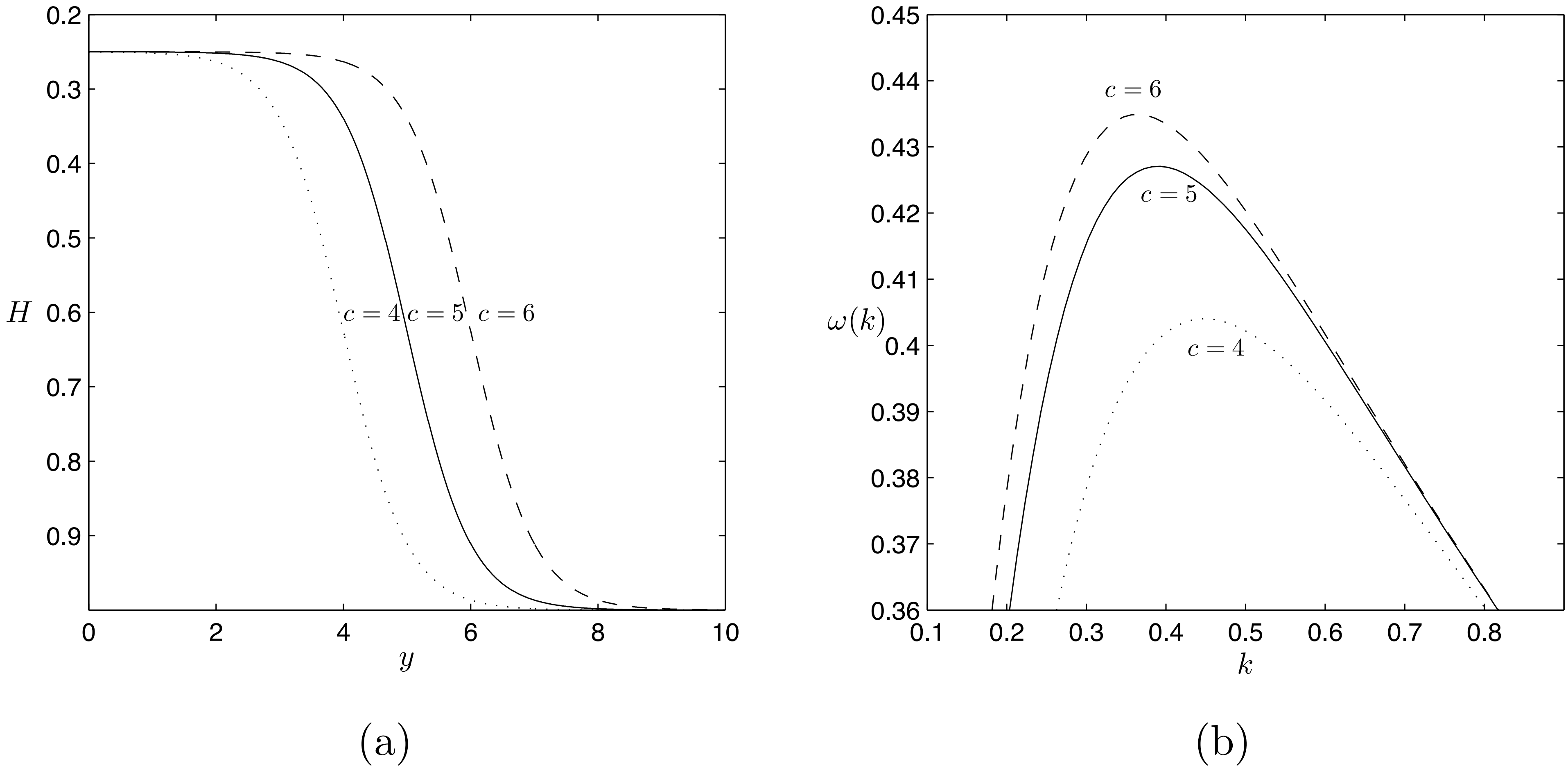
$\mathcal{D}_2(\epsilon)$: difference between the computed value (2D spectral) of $(b/\omega)^2$ for curvature

$$\hat{\delta}(\xi) = (\Delta/2)\tanh\xi, \qquad \hat{\delta}_\xi(\xi) = (\Delta/2)\,\mathrm{sech}^2\,\xi,$$

$\Delta = \pi/2$ and $b = 2$, and the sum of the first three terms of asymptotics (explicit).

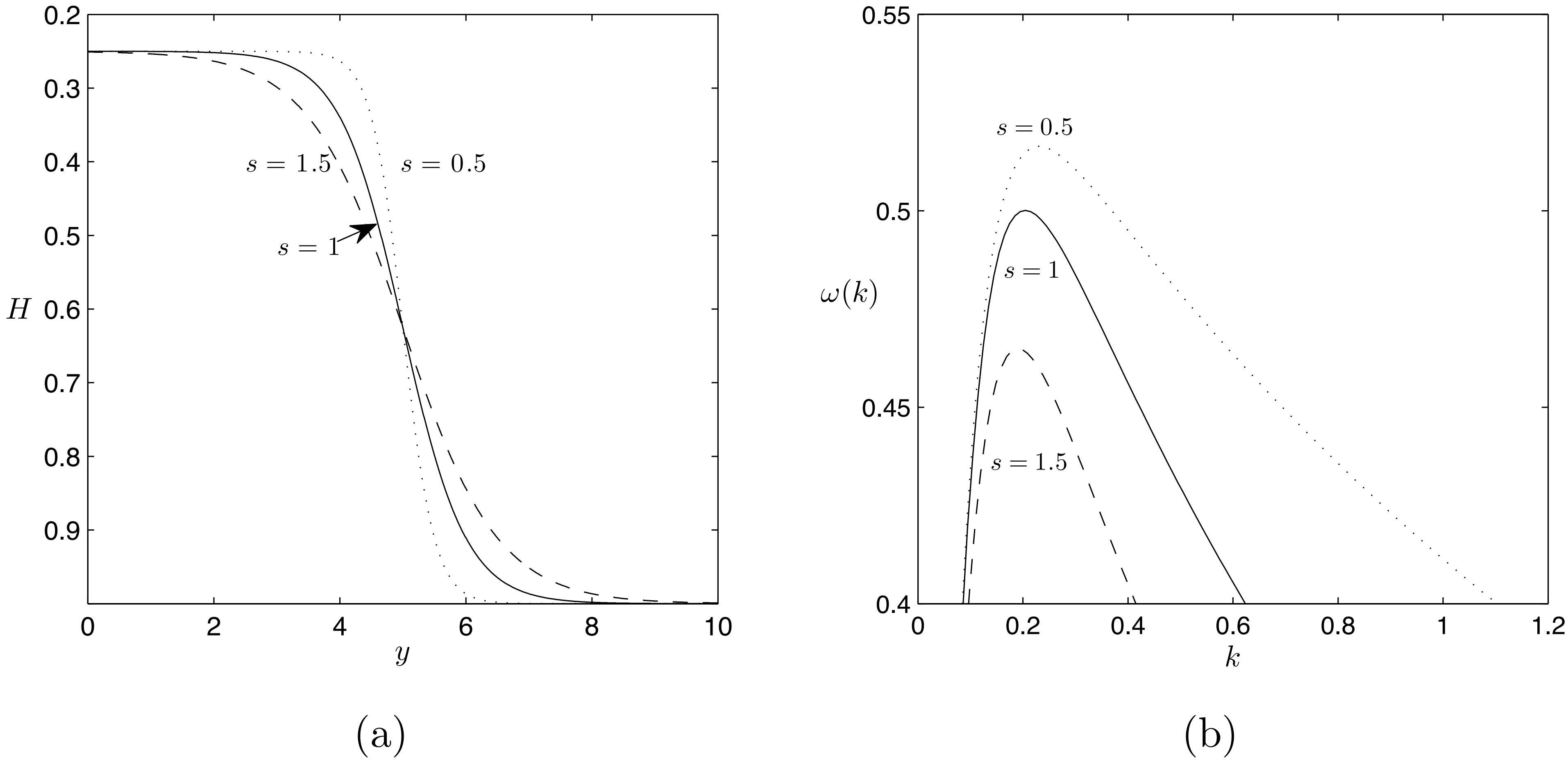
Dotted line slope =4 (J., Rodney & Kaoullas 2011)

Shelf-break to shoreline distance



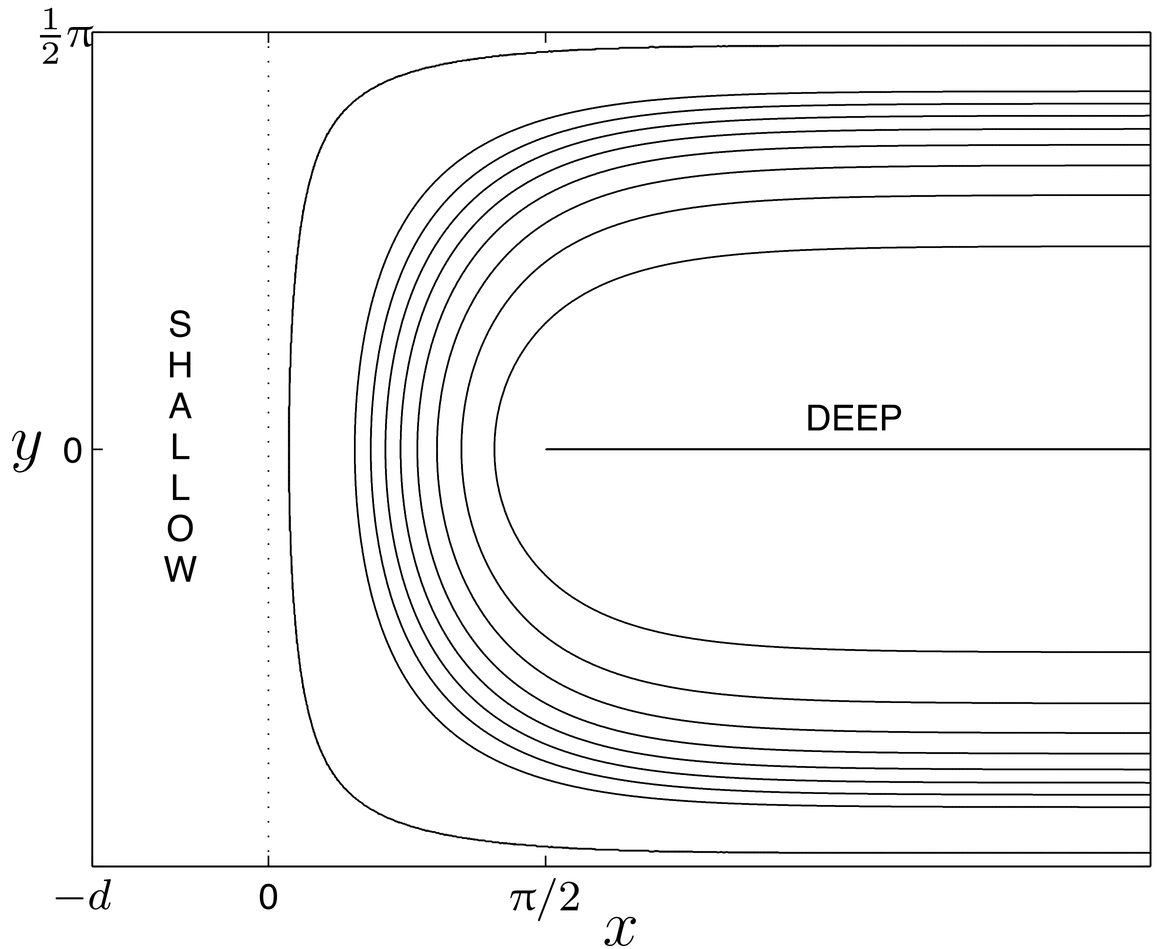
(a) Depth profiles and (b) dispersion curves of the first propagating mode for shelf breaks at $c = 4$ (dotted line), $c = 5$ (solid line) and $c = 6$ (dashed line). Here $s = 1$ and $H_0 = 0.25$.

Shelfbreak steepness



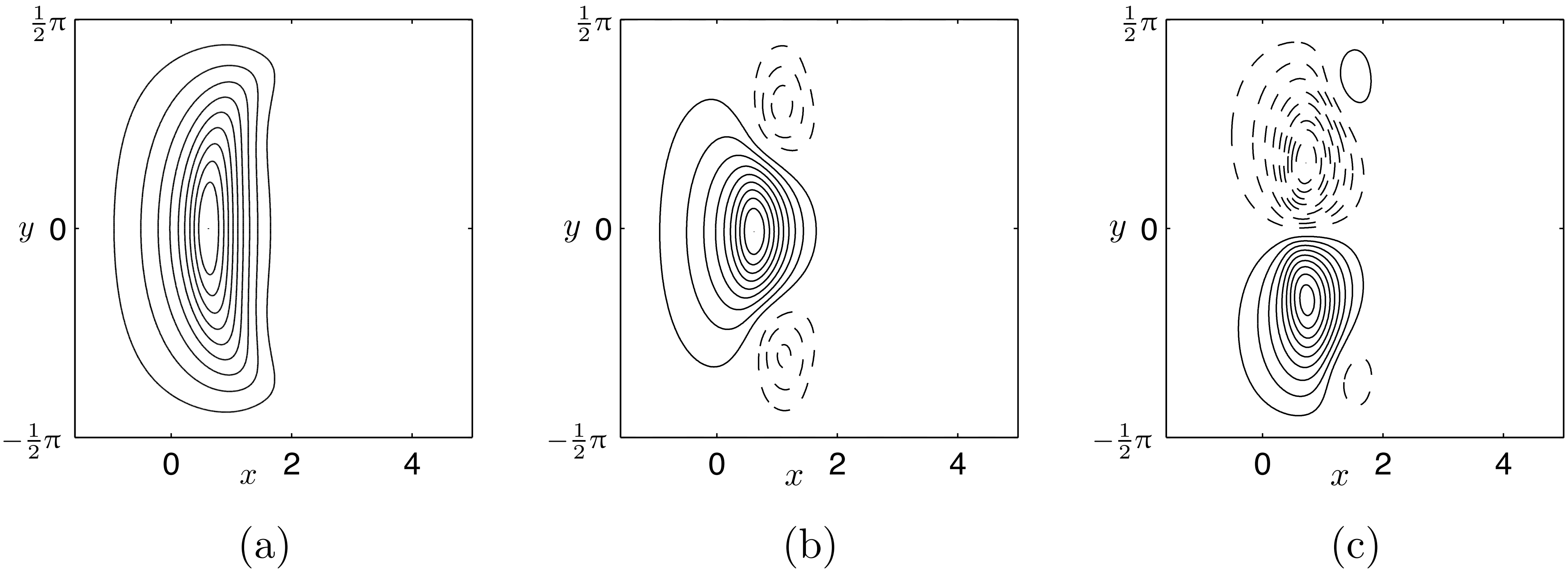
(a) Depth profiles and (b) dispersion curves of the first propagating mode for shelf widths $s = 0.5$ (dotted line), $s = 1$ (solid line) and $s = 1.5$ (dashed line). Here $c = 5$ and $H_0 = 0.25$.

Lake-end isobaths



Isobaths (at 0.1 intervals) for the lake-end model with $|y| \leq \pi/2$, $x \geq -d$. Here $d = 1$, $H_0 = 0.1$ and the maximum depth is 1.

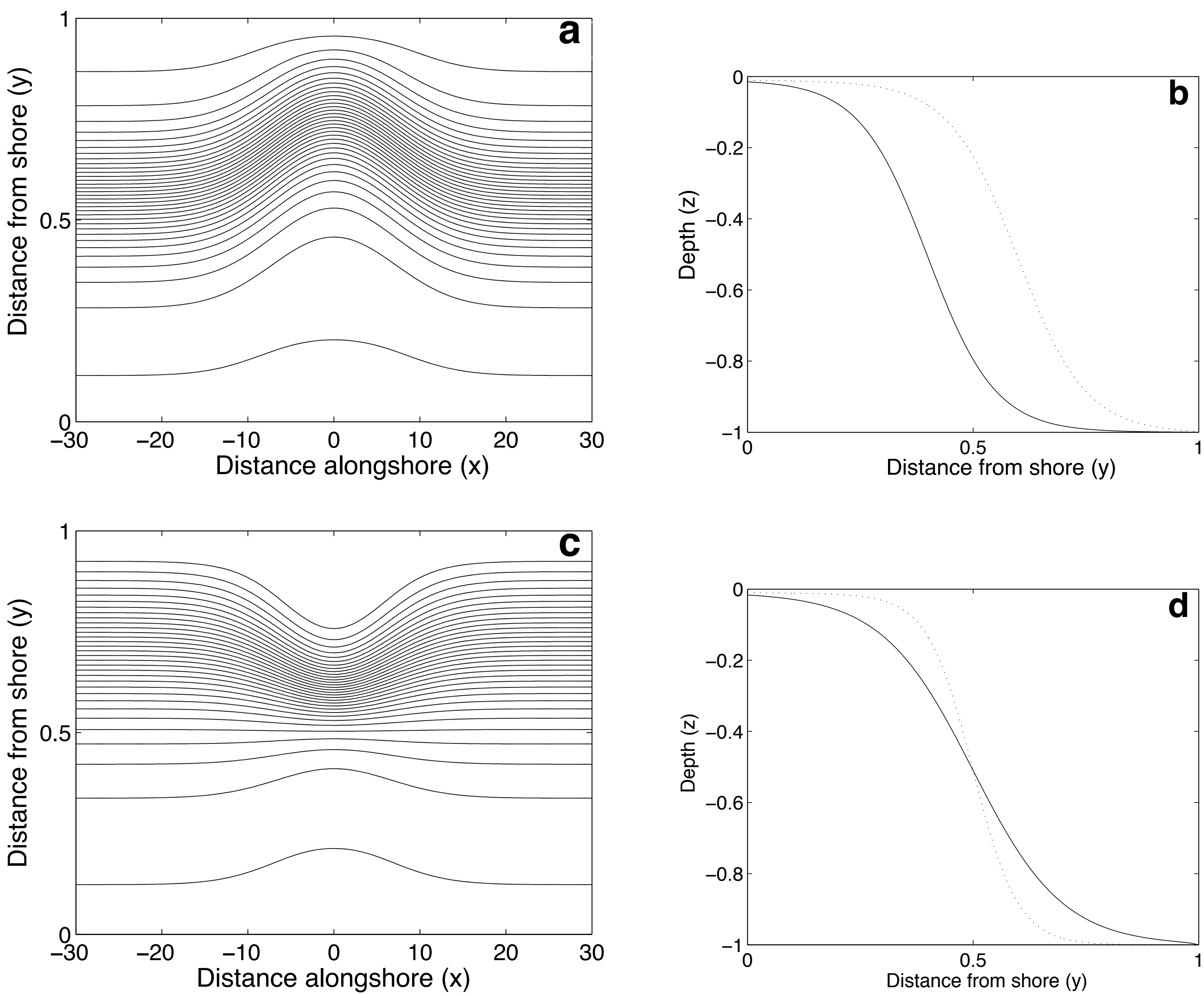
Fundamental lake-end mode



The absolute value (a), and real (b) and imaginary (c) parts, of the streamfunction ψ for the fundamental trapped mode. The contours are at equal intervals with positive values solid and negative values dashed. Here $N = 30$, $M = 72$, $s = 2$, $d = 1.6$ and $L = 9$.

(J. & Kaoullas 2011)

Coastal geometry



Stratified equations

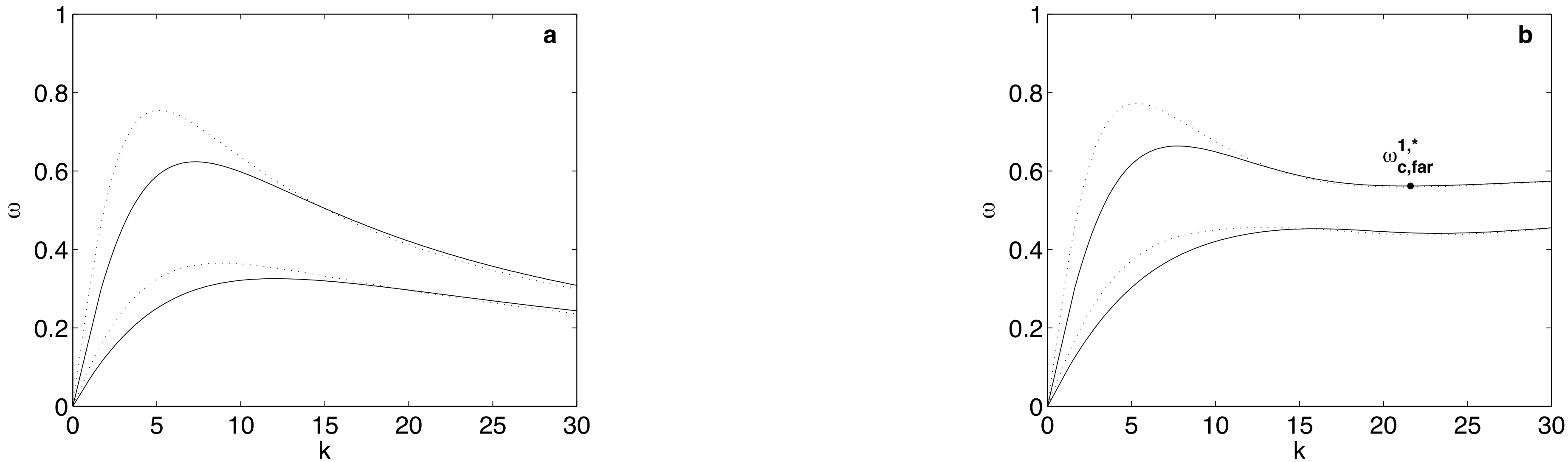
$$
\begin{aligned}
-\mathrm{i}\omega u - v &= -p_x, \\
-\mathrm{i}\omega v + u &= -p_y, \\
-p_z - \rho &= 0, \\
-\mathrm{i}\omega\rho - b^2 B^2 w &= 0, \\
u_x + v_y + w_z &= 0.
\end{aligned}
$$

$$
\begin{aligned}
v &= 0, && (y = 0), \\
w &= -u\hat{h}_x - v\hat{h}_y, && (z = -\hat{h}(x, y),\ y \le 1), \\
p_z &= 0, && (z = 0), \\
\nabla p &\to 0, && x^2 + y^2 \to \infty.
\end{aligned}
$$

Slowly-varying shelf geometry

- Slow variation: $\hat{h}(x, y) = h(\varepsilon x, y) = h(\xi, y)$ where $\varepsilon \ll 1$.
- Classical LGJWKBK ansatz:
  $p(\xi, y, z) \sim \exp(\mathrm{i}S(\xi)/\varepsilon) \sum_{j=0}^{\infty} \varepsilon^j \psi_j(\xi, y, z)$.
- Zero order: x fixed (ξ a parameter). Problem in (y, z) with eigenvalue the local longshore wavenumber $k(\xi)$.

Shelfbreak effects



Dispersion curves for modes 1 and 2 over the far shelf break (dashed line) and close shelf break (solid line) at uniform stratification with (a) $B = 0.01$ and (b) $B = 0.2$.

Localised modes

- $S'(\xi)$ corresponds to the (two) roots $k^{\pm}(\xi)$ for the wavenumber.
- Writing $S^{\pm}(\xi) = l(\xi) \pm m(\xi)$ gives

$$l(\xi) = \frac{1}{2}\int^{\xi} [k^{+}(\xi') + k^{-}(\xi')]\, d\xi', \qquad m(\xi) = \frac{1}{2}\int_{\xi_0}^{\xi} [k^{+}(\xi') - k^{-}(\xi')]\, d\xi'.$$

- $m(\xi)$ gives direction of propagation of the wave envelopes, giving the forward and backward propagating modes S^{+} and S^{-}
- $l(\xi)$ is the (fast) phase of each wave, the same for both modes
- $m(\xi)$ vanishes when $k^{+} = k^{-}$, at $\xi = \pm\xi_c$ (say) for symmetric perturbations (group velocity vanishes). For $|\xi| > \xi_c$ waves evanescent.
- Matching determines frequency through

$$\frac{1}{2\varepsilon}\int_{-\xi_c}^{\xi_c} [k^{+}(\xi') - k^{-}(\xi')]\, d\xi' \sim (n + \frac{1}{2})\pi + O(\varepsilon), \quad n = 0, 1, 2, \cdots ,$$

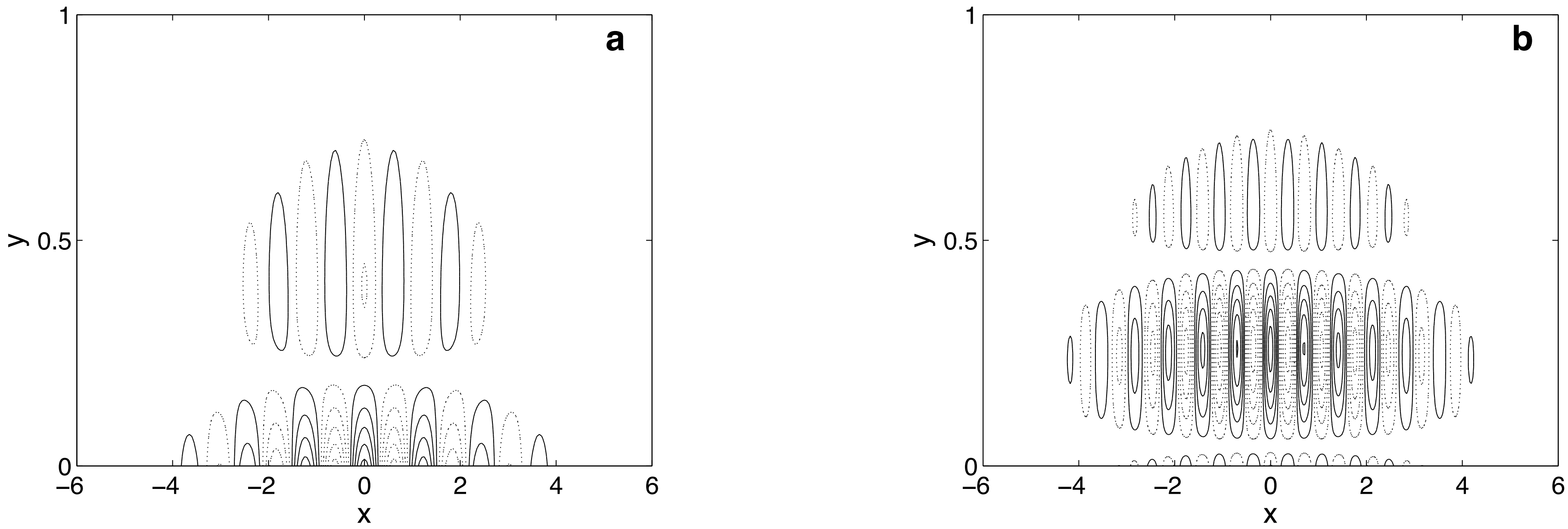
Numerical solutions

- JWKB:
  - At each ξ station, discretize 2D spectral.
  - For ξ increasing from $\xi = 0$, compute $k^{\pm}$ by inverse iteration from previous station (extremely fast)
  - 10 stations sufficient for 10 sig. fig. for ω
- 3D numerical:
  - Spectral 3D – Chebyshey vertical, Laguerre offshore, Hermite alongshore.
  - Inverse iteration using JWKB very fast.

Eigenvalues

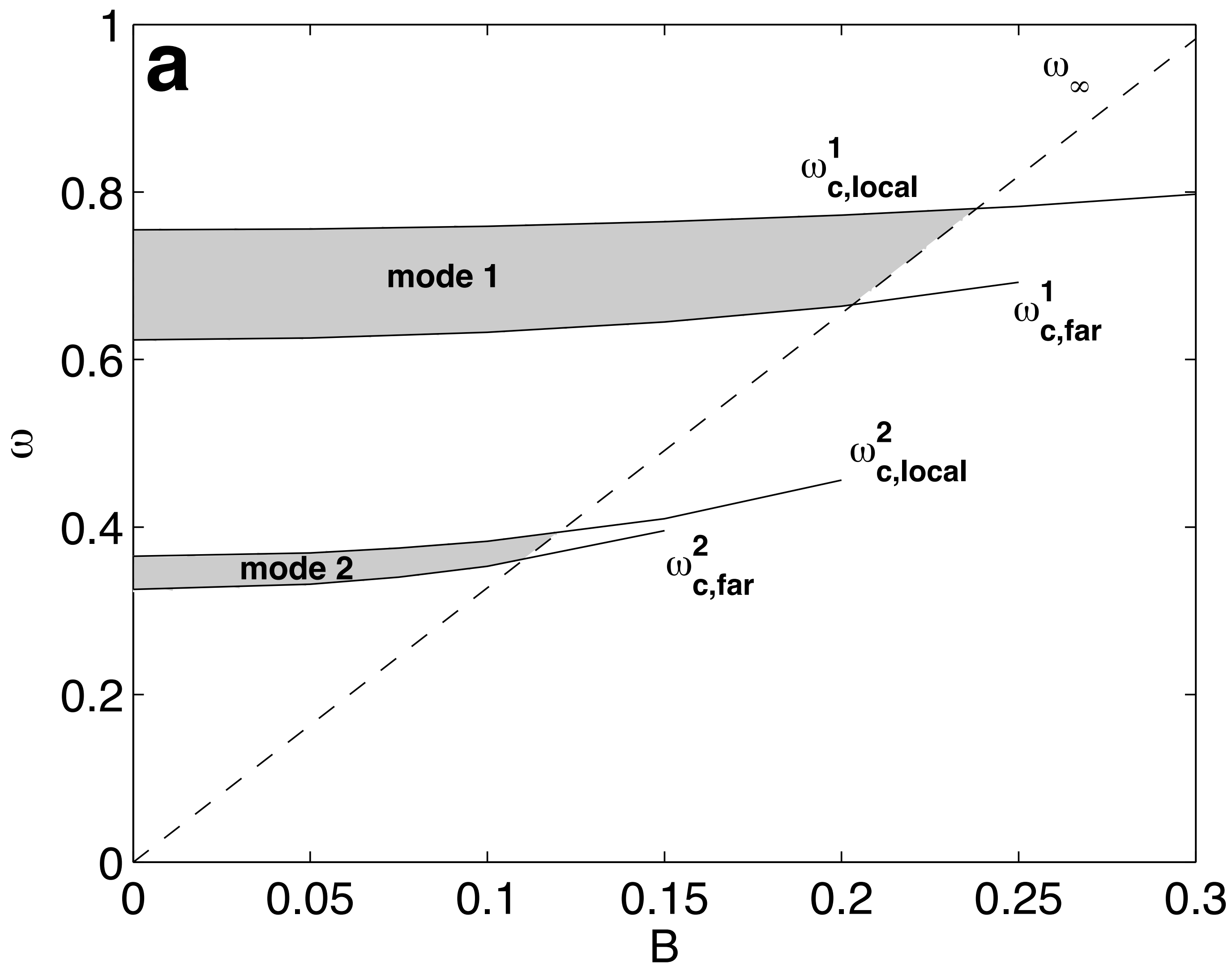
n	m	B	$\omega_{n,m}^{A}$	$\omega_{n,m}^{N}$	Error (%)
0	1	0.01	0.7518365	0.7519881	0.0201
1	1	0.01	0.7454791	0.7456205	0.0190
2	1	0.01	0.7393040	0.7394385	0.0182
0	1	0.15	0.7615593	0.7616983	0.0182
1	1	0.15	0.7555876	0.7557198	0.0175
2	1	0.15	0.7497965	0.7499211	0.0166
0	2	0.01	0.3647694	0.3647729	0.0009
1	2	0.01	0.3640451	0.3640447	0.0001

3D modes



(Rodney & J. 2011)

Regime diagram



- One wave
- Three waves